Directorate of Distance Education

UNIVERSITY OF JAMMU

JAMMU

SELF LEARNING MATERIAL

B.A. SEMESTER - VI

SUBJECT : STATISTICS — **UNIT - I to V**

COURSE NO. : ST - 601 (THEORY) — **LESSON NO. – 1 to 21**

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COURSE CO-ORDINATOR

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STATISTICS

VI Semester (B.A./B.Sc)

Examination to be held in the years May 2017, 2018 and 2020 & 2021

Paper Code: ST 601(Theory) **Title: APPLIED STATISTICS-II**

Duration: 3 Hours **Max Marks: 100**
Credit: 4 Credit **Theory Examination: 80**
Internal Assessment: 20

Objective: The main objective of this course is to provide knowledge to the students about statistical quality control and computational techniques of Numerical Analysis and LPP.

Unit- I

Indian applied statistical system; Present official statistical system in India, Method of collection of official statistics. Role and Functions of MOSPI, CSO, NSSO and Directorate of Economics and Statistics of J&K Government. Importance of statistical methods in industrial research and practice, types of inspections, determination of tolerance limits.

Unit- II

General theory of control charts, cause of variation in quality, control limits. sub-grouping, summary of out of control and criteria charts for attributes, np-chart, p-chart, c-chart. charts for variables: mean and Range - Charts, design of mean and Range charts versus P charts, process capability studies.

Unit- III

Principle of acceptance sampling:- Problem of lot tolerance, stipulation of good and bad lots, producers and consumer risks, single and double sampling plans, their OC functions, concept of AOL, LTPD, AOQ L average amount of inspection and ASN function. Rectifying inspection plan, Sampling Plan, Concept of $(6-\sigma)$ limits.

Unit- IV

Computational technique: difference table and method of interpolation. Newton and Lagrange's method of interpolation, divided difference, numerical differentiation and integration, Trapezoidal rule, Simpson 1/3 and 3/8 rule.

Unit-V

Linear Programming: elementary theory of convex set, definition of general LPP. Formulation problem of LPP, Example of LPP, problem occurring in various fields, graphical and simplex method of solving an LPP, artificial variable, duality of LPP.

Note for paper setting:

The question paper will contain three Sections. Section A will contain compulsory ten very short answer type questions of-1 mark each. Section B will contain 7 short answer type questions of 5 marks each, at least one question from each unit and the student has to attempt any five questions. Section C will contain 10 long answer type questions, two from each unit, of 9 marks each and the student has to attempt five questions selecting one from each unit.

Internal Assessment (Total Marks: 20)

20 marks for theory paper in a subject reserved for internal assessment shall be distributed as under:
Two written Assignments/Project reports: 10 mark each

Books Recommended

1. Brownlee K.A. (1960): Statistical Theory and Methodology in Science and Engineering. John Wiley and Sons

2. Grant E.L. (1964): Statistical quality control. McGraw Hill.

3. Duncan A.1. (1974); Quality control and Industrial Statistics. Taraporewala and soils.

4. Gass S.l. (1975) Linear Progamming methods and applications. Mc Graw Hill.

5. Rajaraman, V (1981): Computer Oriented Numerical Methods. Prentice hall:

6. Sastry S.S. (1987) : Introductory methods of numerical analysis. Prentice hall

7. Taha H.A. (1989): Operation Research: An Introduction. Macmillan Publishing

Additional References

8. Broker H.A. and Liberman G.T. (1962): Engineering Statistics. Prentice Hall.

9. Cowden D.I. (1960): Statistical Methods in Quality Control. Asia Publishing Society.

10. Gavin W.W. (1960): Introduction to linear programming. Mc Graw Hill.

11. Mahajan M.200I): Statistical Quality Control. Dhanpat Rai and Co. (P)Ltd.

12.Rao S.S.(1984); Optimization Theory and Applications. Wiley Eastern.

STATISTICS

VI Semester (B.A./B.Sc)

Examination to be held in the years May 2017, 2018 and 2020 & 2021

Paper Code: ST-601 (Practical) **Title: STATISTICAL COMPUTING-VI**

Max Marks: 50
External Assessment: 25
Internal Assessment: 25

Objectives: The objective of the course is to expose the students to the real life application of SQC and Computational techniques.

Syllabus: There shall be atleast fifteen computing exercises covering the applications based on the entire syllabus of course ST-601(Theory)

Distribution of internal Assessment (25 Marks)

(i) I Assessment: 06 Marks

(ii) II Assessment: 06 Marks

(iii) Class Test : 08 Marks

(iv) Attendance: 05 Marks

UNIT-1

Lesson - 1

1.1 Objectives

1.2 Introduction to Statistical system.

1.3 Statistical system in India, past and present.

1.4 Role and functions of MOSPI.

1.5 Summary and further suggested readings.

Lesson - 2

2.1 Objectives

2.2 Introduction to CSO, its role in Indian official statistical system.

2.3 NSSO and its functions.

2.4 Directorate of Economics and Statistics, J&K Government, its functions.

2.5 Summary and further suggested readings.

Lesson - 3

3.1 Objectives

3.2 Introduction to Quality.

3.3 Meaning and dimensions of Quality.

3.4 Quality engineering terminology.

3.5 Importance of statistical methods in industrial research and practice.

3.6 Summary and further suggested readings.

Lesson - 4

4.1 Objectives

4.2 Measurement and determination of quality factors.

4.3 Count and measurement and visual gauging.

4.4 Types of inspection.

4.5 Determination of natural tolerance limits.

4.6 Summary and further suggested readings.

1.1 Objectives

Objectives of this lesson are :-

- To provide a overview of the statistical systems prevailing around us
- To define the meaning of official statistical system
- To introduce the working of statistical system in India.

1.2 INTRODUCTION TO STATISTICAL SYSTEM

A national statistical system becomes necessary for organizing the collection, compilation and publication of a series of statistics on all important aspects of national life on regular basis. Besides carrying on research for developing proper procedures of collection and analysis of data, the national system has to co-ordinate the work of various statistical offices of the country.

A statistical system may be examined from a variety of angles, perhaps the most important consideration is the degree of centralization involved in the system and ion the basis, of these criteria, the following broad types of statistical system may be distinguished.

(a) Completely decentralized system: Under this set-up, no system really exists, having no co-coordinating agency and the quality of data varies from subject to subject.

(b) System with minimum of coordination: In this system there are several offices at national level, each responsible for collection, compilation and publication of data in its own area.

(c) System decentralized with minimum of co-ordination. Here different offices are responsible for collection, compilation and publication of data, but there is an agency to coordinate their activities which ensures standard definitions, uniform coverage and comparable time periods are adopted.

(d) System with a central office for general statistics and a coordinating agency. Here certain types of statistics may be derived as a by-product of routine activities of various departments; there are others that have to be collected by special organizations. Under this set-up a central office is entrusted with the collection of latter type of data, quite often this office also functions as an agency to coordinate the work of various departments.

(e) Centralized system. Here a central office collects, compiles and publishes statistics relating to social and economic matters. It also takes upon itself the task of compiling administrative and specialized data collected by various departments. This system leads to a considerable economy in tabulation.

1.3 STATISTICAL SYSTEM IN INDIA, PAST AND PRESENT

Statistical information about a place, region, state or nation is necessary for running the administration efficiently.

India has a long historical tradition of collection and use of various kinds of statistics. Its origin dates back to the period of Ramayana when statistics, regarding employment and taxes, were collected during the rule of Raja Bharat. Kautilya Arthashastra also throws light on the collection machinery of statistical information regarding land, area, agricultural production, population, taxes etc. Kautilya's Arthasastra (321-296B.C.), one of the greatest treatise of economics, indicates a system of census and data collection relating to agriculture, population and other economic activities, covering villages and towns. In addition, the concept of crosschecking and validation by independent agents was very much part of the data collection system.

During the Moghul period, evidence and application of statistical knowledge was prominent in Ain-i-Akbari by Abul Fazal. Documentary evidence includes the system of legalised measurements, land classification and crop yields by season, etc. The system of land tenure and land revenue, followed during the Moghul period, had enough empirical basis.

The statistical system was strengthened during the British period. During this period, the statistical development was geared towards administration, tax collection, revenue, trade and commerce and related activities as might be expected.

Thus the main documents Systematic data collection started in India only with the advent of British rule and first population census was taken in 1871-72, however other modes of data collection started much earlier for instance in 1848, the first census relating to the area and revenue of each in North-West Provinces was released. In 1853, the department released the first series of statistical papers on India. Impressed by the trend in statistical activities, the Secretary of State ordered the Governor-General in Council to prepare a 'comprehensive and coordinated scheme of statistical survey' for each of the twelve great provinces of the then British India and Dr. W.W. Hunter was appointed as Director-General of Statistics in India in 1869, who can perhaps be regarded as the original precursor to the Chief Statistician of India today. The operation of a decennial census for the whole country started in 1881 and is continuing ever since. The report on the Census of British India taken in 1881 was published in three volumes and till the attainment of independence there was no government agency for systematic collection of data.

POST-INDEPENDENCE PERIOD

Indian Government has a federal set up as far as the administration of the country is concerned. Hence, there is a combined responsibility of the states as well as the centre for internal matters. In view of this, all the jobs are taken either by state government or by the central government or by both. Different categories of works are covered by different ministries of the states or the centre. For a systematic organisation and the collection of statistical information, Government of India established different departments at the centre and state levels. The Cabinet Secretariat has a Department of Statistics. Steps were taken for the economic development of the country through successive five years plans; an urgent need was felt for placing the system of data collection on sound footing. Professor P.C. Mahalanobis, who is regarded as a pioneer in both theoretical and professional statistics, was appointed as the first statistical adviser to the Cabinet, Government of India in January 1949. He was the architect of the statistical system of independent India. Professor P. V. Sukhatme, as Statistical Adviser to the Ministry of Agriculture, was responsible for the development of Agricultural Statistics. A Central Statistical unit was set-up in the cabinet secretariat in1949.

This was expanded into present Central Statistical Organization (CSO) in 1951.The purpose was to effect the coordination of the work of

(i) The statistical units in various central government departments.

(ii) The Statistical Bureau of the various state governments.

At present in India we have a broadly decentralized statistical system in which CSO, with its headquarters in New Delhi acts as apex or advisory and coordinating body. The structure of Indian statistical system, in away, a consequence of division of responsibilities between the Union and State Governments under a federal constitution and the needs of individual ministries for statistics pertaining to their own administrative functions. Under the constitution, foreign trade, railways, banking and currency, population, telegraph etc, are central subjects. The Government of India bears full responsibility and cost of collection on these subjects. But there are certain subjects on which the state and the central governments operate simultaneously to meet their requirements.

But even where the States have their primary duty of data collection , central Government acts as coordinating agency for compilation and publication of data on an all India basis through CSO. The CSO can, and dose, issue directives to the State in order to bring about uniformity in data collected at the state level.

National Sample Survey Organisation (NSSO) set up in January, 1950. Another wing is the Central Statistical Organisation (CSO) set up in May, 1951. These two are important statistical wings of the Government of India

ORGANISATIONS : The important organisations which are engaged in one or the other way for the collection tabulation and analysis of data can be categorised as central state and other organisation.

STATISTICAL SYSTEM AT THE CENTRE

The collection of statistics for different subject-specific areas, like agriculture, labour, commerce, industry, etc. vests with the corresponding administrative ministries. More often than not, the statistical information is collected as a by-product of administration or for monitoring the progress of specific programmes. Some of the ministries, like Agriculture, Water Resources, Health, etc. have full-fledged statistical divisions, while most others have only a nucleus cell. Large-scale statistical operations like the Population Census, Annual Survey of Industries, Economic Census, etc. are generally centralised, and these cater to the needs of other ministries and departments, as well as State Governments. In important ministries, officers of the Indian Statistical Service (ISS) and subordinate statistical staff perform the statistical functions. The Central Statistical Organisation (CSO) in the Ministry of Statistics and Programme Implementation (MoS&PI) is the nodal agency for a planned development of the statistical system in the country and for bringing about coordination in statistical activities among statistical agencies in the Government of India and State Directorates of Economics and Statistics. Further details about the coordinating role of CSO along with its other activities have been given in the Report elsewhere. Some of the most important of these organizations are Central statistical organisation (CSO), national sample surveyorganisation (NSSO), Office of the registrar general of India (RGI), Directorate of Economics & Statistics (D E S).

Statistical Organizations in the States and Union territories: Statistical offices in the states (UT's) are of more recent origin than those at centre. The Statistical System in the States is similar to that at the Centre. It is generally decentralised laterally over the Departments of the State Government, with major Departments, such as, agriculture or health, having large statistical divisions for the work of departmental statistics. State Statistical Bureaus are now established in all the states. The bureau in a state has its functions of coordination of the statistics collected by different departments of the state government and the publication of the abstract assembling all essential statistical series. It also maintains a liaison between statistical units in the state departments on one hand and the C.S.O on the other. Most of them participate at least on a matching sample basis in the national Sample Survey Programme, and some of them carry out an Annual Survey of Industries for factories not covered by the ASI of the NSSO. Generally, the States do not have a common statistical cadre.

Non Government Statistical Organizations: The following are the some of the main N.G.O's working in the country

(i) ISI Calcutta

(ii) National council of applied Economics research

(iii) Institute of Economic Growth

(iv) Gokhle Institute of Economics and Politics, Pune

(v) Tata Institute of Social Sciences

(vi) Statistics Deptt of R.B.I

(vii) Universities of the country.

1.4 ROLE AND FUNCTIONS OF MOSPI

MINISTRY OF STATISTICS AND PROGRAMME IMPLEMENTATION (MOSPI)

The Ministry of Statistics and Programme Implementation came into existence as an Independent Ministry on 15.10.1999 after the merger of the Department of Statistics and the Department of Programme Implementation. The Ministry has two wings, one relating to Statistics and the other Programme Implementation. The Statistics Wing called the National Statistical Office (NSO) consists of the Central Statistical Office (CSO), the Computer Centre and the National Sample Survey Office (NSSO).

The Programme Implementation Wing is headed by Minister of State (Independent Charge). At the executive level, it is headed by Secretary to Government of India, who is also the Chief Statistician of India has three Divisions namely,

(i) Twenty Point Programme

(ii) Infrastructure Monitoring and Project Monitoring

(iii) Member of Parliament Local Area Development Scheme.

Besides these two wings, there is National Statistical Commission created through a Resolution of Government of India (MOSPI) and one autonomous Institute, viz., Indian Statistical Institute declared as an institute of National importance by an Act of Parliament.

The Ministry of Statistics and Programme Implementation attaches considerable importance to coverage and quality aspects of statistics released in the country. The statistics released are based on administrative sources, surveys and censuses conducted by the Centre and State Governments and non-official sources and studies. The surveys conducted by the Ministry are based on scientific sampling methods. Field data are collected through dedicated field staff. In line with the emphasis on the quality of statistics released by the Ministry, the methodological issues concerning the compilation of national accounts are overseen Committees like Advisory Committee on National Accounts, Standing Committee on Industrial Statistics, Technical Advisory Committee on Price Indices. The Ministry compiles datasets based on current data, after applying standard statistical techniques and extensive scrutiny and supervision.

The National Statistical Organization (NSO) consists of two offices viz. Central Statistics Office (CSO) and National Sample Survey Office (NSSO). The CSO and NSSO of the National Statistical Organization are headed by the respective Director Generals.

The Programme Implementation Wing of the Ministry, headed by Additional Secretary, consists of three Divisions viz. Member of Parliament Local Area Development Division (MPLAD Division), Infrastructure and Project Monitoring Division (IPMD), and Twenty Point Programme Division (TPP Division).

Besides National Statistical Organization and Programme Implementation Wing, the Ministry also has an Administrative Wing, responsible for establishment, housekeeping and vigilance matters besides cadre management of Indian Statistical Service (ISS), and Subordinate Statistical Service (SSS).

NSO IS MANDATED WITH THE FOLLOWING RESPONSIBILITIES:-

(i) Acts as the nodal agency for planned development of the statistical system in the country, lays down and maintains norms and standards in the field of statistics, involving concepts and definitions, methodology of data collection, processing of data and dissemination of results;

(ii) Coordinates the statistical work in respect of the Ministries/Departments of the Government of India and State Statistical Bureaus (SSBs), advises the Ministries/ Departments of the Government of India on statistical methodology and on statistical analysis of data;

(iii) Prepares national accounts as well as publishes annual estimates of national product, government and private consumption expenditure, capital formation, savings, estimates of capital stock and consumption of fixed capital, as also the state level gross capital formation of supra-regional sectors and prepares comparable estimates of State Domestic Product (SDP) at current prices;

(iv) Maintains liaison with international statistical organisations, such as, the United Nations Statistical Division (UNSD), the Economic and Social Commission for Asia and the Pacific (ESCAP), the Statistical Institute for Asia and the Pacific (SIAP), the International Monetary Fund (IMF), the Asian Development Bank (ADB), the Food and Agriculture Organisation (FAO), the International Labour Organisation (ILO), etc.

(v) Compiles and releases the Index of Industrial Production (IIP) every month in the form of 'quick estimates'; conducts the Annual Survey of Industries (ASI); and provides statistical information to assess and evaluate the changes in the growth, composition and structure of the organised manufacturing sector;

(vi) Organises and conducts periodic all-India Economic Censuses and follow-up enterprise surveys, provides an in-house facility to process the data collected through various socio-economic surveys and follow-up enterprise surveys of Economic Censuses;

(vii) Conducts large scale all-India sample surveys for creating the database needed for studying the impact of specific problems for the benefit of different population groups in diverse socio-economic areas, such as employment, consumer expenditure, housing conditions and environment, literacy levels, health, nutrition, family welfare, etc;

(viii) Examines the survey reports from the technical angle and evaluates the sampling design including survey feasibility studies in respect of surveys conducted by the National Sample Survey Organisation and other Central Ministries and Departments;

(ix) Dissemination of statistical information on various aspects through a number of publications distributed to Government, semi-Government, or private data users/

agencies; and disseminates data, on request, to the United Nations agencies like the UNSD, the ESCAP, the ILO and other international agencies;

(x) Releases grants-in-aid to registered Non-Governmental Organizations and research institutions of repute for undertaking special studies or surveys, printing of statistical reports, and financing seminars, workshops and conferences relating to different subject areas of official statistics.

PROGRAMME IMPLEMENTATION WING HAS THE FOLLOWING RESPONSIBILITIES:-

(i) Monitoring of the Twenty Point Programme (TPP);

(ii) Monitoring the performance of the country's eleven key infrastructure sectors, viz., Power, Coal, Steel, Railways, Telecommunications, Ports, Fertilizers, Cement, Petroleum & Natural Gas, Roads and Civil Aviation;

(iii) Monitoring of all Central Sector Projects costing Rs.20 crore and above; and

(iv) Monitoring the implementation of Member of Parliament Local Area Development Scheme (MPLADS).

VISION: To be the finest and most creative National Statistical System in the world; and to effectively monitor the programmes and projects for ensuring efficient use of national resources.

MISSION

1. To make available reliable and timely statistics and to undertake regular assessment of data needs for informed decision making;
2. To cater to the emerging data needs in a dynamic socio-economic context, to reduce respondent burden and to avoid unnecessary duplication in data collection and publication;
3. To adopt and evolve standards and methodologies for statistics generated by various elements of the National Statistical System and to steer its development for further improvement and bridging data gaps;

4. To ensure and strengthen trust and confidence of all stake holders in the National Statistical System by maintaining confidentiality of data providers and promoting integrity and impartiality of all elements of official statistics ;

5. To provide leadership and coordination to ensure harmonious, efficient and integrated functioning of all the elements of the National Statistical System;

6. To continue to asses skill requirement, and develop human resource capacity at all levels of the statistical system;

7. To participate and contribute actively in all international initiatives and to support development of Statistical Systems around the world ;

SERVICES

1. To provide reliable, timely and credible data/ statistics to policy makers in the central and state governments, for monitoring, planning and policy formulation, and also to our other clients as per the National Policy on Dissemination of Statistical Data ;

2. To conduct research on measurement and analysis of socio-economic issues and other emerging areas, and make available the results in the form of reports/publications to users both in government and outside ;

3. To conduct studies on the implementation of government projects and programmes ;

4. To release Publications as per the 'Publication Calendar' placed on the website of the Ministry;

5. To make available select data to users at one place through 'Data Warehouse';

6. To provide advise to the Ministries /Departments of Government of India and State Governments on statistical matters like methodologies, concepts, definitions, standards, classifications, etc ;

7. To provide training for Central/State Governments, international agencies and others as per the 'Training Calendar" placed on the website of the Ministry, and also to conduct special training programmes as and when necessary ;

1.5 SUMMARY AND FURTHER SUGGESTED READING

The main objective of this lesson was to provide an overview of the statistical systems prevailing around us and define the meaning of official statistical system. In addition to this, the working of statistical system in India was also discussed. Indian Government has a federal set up as far as the administration of the country is concerned. Hence, there is a combined responsibility of the states as well as the centre for internal matters.

At present in India we have a broadly decentralized statistical system in which CSO, with its headquarters in New Delhi acts as apex or advisory and coordinating body. The structure of Indian statistical system, in a way, a consequence of division of responsibilities between the Union and State Governments under a federal constitution and the needs of individual ministries for statistics pertaining to their own administrative functions.

FURTHER SUGGESTED READING

1. Goon, Gupta and Das Gupta:Fundamentals of Statistics,World Presss

2. Various publications of RGI,NSSO,CSO,MOSPI

SELF ASSESSMENT QUESTIONS

1. Define official statistical system, discuss the working of official statistical system in India.

2. Elaborate the role of agencies responsible for collection of official statistical system in India.

3. Discuss the role of MOSPI in India.

2.1 OBJECTIVES

The main objectives of this lesson are :-

- To introduce the students with CSO, its working and role in Indian official statistical system.
- To introduce the students with NSSO and its functioning.
- To introduce the students with DES J&K government, its functions.

2.2 INTRODUCTION TO CSO, ITS ROLE IN INDIAN OFFICIAL STATISTICAL SYSTEM

CENTRAL STATISTICAL ORGANISATION (CSO)

With a view to coordinating statistical activities of the different ministries of the Government of India and the State Governments and the evolving of statistical standards, the CSO was set-up by the government of India in 1951 as a part of Cabinet Secretariat having coordinating and advisory functions for statistical activities of the various central and state departments; it provides national statistics to United Nations and its specialized agencies and brings out publications

Most of the Central government Ministries have their own statistical units which collect and use statistics in their respective fields. Besides these units there are organizations established by the government specifically for the purpose of collection, compilation and publication of data Central Statistical Organisation (CSO) is one of these organizations

The scope and duties of CSO has widened following the transfer of National Income Unit from Ministry of Finance to CSO in 1954 and transfer of Directorate of Industrial

Statistics in 1957 to CSO. In recent years a separate unit has been established to attend statistical work related to five years plans in collaboration with Planning Commission. It was put under the Ministry of Planning in 1969 and a separate Ministry of Statistics and Program Evaluation in 1999. As a part of its advisory and coordinating functions, the CSO has been engaged in setting and improving the standards regarding concepts, definitions classifications and methodology of data collection.

The responsibilities of CSO include coordinating statistical activities and liaison with the Central Government Departments, State Governments and International Agencies; preparation of national accounts; conducting Annual Survey of Industries, Economic Censuses and their Follow-up Enterprise Surveys; constructing IIP and consumer price indices for urban non-manual employees; compiling Social Sector Statistics; imparting training in official statistics; formulating a Five Year Plan programme relating to development of statistics in the States and Union Territories; disseminating various statistical information including those relating to social and environment statistics; undertaking periodic revision of National Industrial Classification, etc. The CSO is also responsible for periodically conducting the Conference of Central and State Statistical Organisations.

One of the major responsibilities of the CSO is to act as the nodal agency for planned development of the statistical system of the country. The CSO is entrusted with the responsibility not only to coordinate the statistical activities of the Government of India and State Directorates of Economics and Statistics (DESs) but also to lay down and maintain norms and standards in the field of statistics.

1. Coordination of statistical activities at the centre and the state. The CSO, as a coordinating agency, should maintain a pool of eminent experts in different subject areas. This would be useful for getting comments on various statistical matters quickly and also for constituting various committees and working groups on technical matters.

2. Advisory work concerning the statistical matter, particularly standardization of concepts and definitions to maintain uniformity, throughout the country.

3. Collection of statistical data related to planning.

4. Training of statistical personnel.

5. Compilation of national income estimates.

6. To provide statistical data of the nation to the United Nations statistical offices and other international institutions.

7. To plan and coordinate the conduct of the annual survey of industries and publish the results.

8. To attend to the work of International Statistical Institutes (conferences) held in India and abroad.

9. The display of charts and graphs pertaining to the national data which are of administrative interest.

10. Circulation of regular publications.

The CSO through its Industrial Statistics Wing conducts the annual survey of industries and publishes results.

THE MOST IMPORTANT PUBLICATIONS OF CSO ARE

(i) The Statistical Abstract-India (annual)

(ii) The Monthly Abstract of Statistics

2.3 NSSO ITS FUNCTIONS

NATIONAL SAMPLE SURVEY ORGANISATION (NSSO)

The National Sample Survey (NSS), initiated in the year 1950, is a nation-wide, large-scale, continuous survey operation conducted in the form of successive rounds. It was established on the basis of a proposal from Professor P.C. Mahalanobis to fill up data gaps for socio-economic planning and policy-making through sample surveys.

Initially, all aspects relating to the designing of surveys, processing of data and preparation of reports were entrusted to the Indian Statistical Institute (ISI). The then Directorate of NSS in the Government of India had been responsible for carrying out the fieldwork in all areas except the State of West Bengal and Bombay City, where the fieldwork was carried out by the ISI. To get rid of inordinate delay in release of survey results, all aspects of survey work were brought under a single umbrella by setting up the National Sample Survey Organisation (NSSO) under the resolution dated 5th March 1970.

CURRENT STATUS

The NSSO carries out Household and Enterprise Surveys, undertakes the fieldwork for the Annual Survey of Industries, provides technical guidance to the States in respect of the Crop Estimation Surveys besides assessing the quality of primary work done by the State Agencies in area enumeration and yield estimation, prepares the urban frames useful for selection of urban blocks for the surveys and collects price data for rural retail prices as well as selected items consumed by the urban non-manual employees required for the preparation of consumer price indices for agricultural labourers and urban non-manual employees, respectively.

The NSSO has four Divisions namely, the Survey Design and Research Division (SDRD), Field Operations Division (FOD), Data Processing Division (DPD), and Coordination and Publication Division (CPD), with each Division headed by an Additional or Deputy Director General.

The NSS is carried out in the form of successive rounds. A unique feature of the NSS is that all the State and Union Territory Governments except the Union Territories of Andaman and Nicobar Islands, Dadra and Nagar Haveli, and Lakshadweep participate in the programme at least on an equal matching sample basis.

Since its inception in 1950, the NSS has collected data on a large number of subjects of interest. The NSS has completed its 56th Round of survey in June 2001. The subjects of enquiry were unregistered manufacture, household consumer expenditure and employment-unemployment. The fieldwork of the NSS 57th Round (covering household consumer expenditure, employment-unemployment and most of the non-agricultural economic activities other than manufacturing and trade) is in progress and is likely to be completed by the end of June 2002.

Summing up, it was started in 1950 as a multipurpose continuing survey for collecting information on all aspects of Indian Economy; it caters the needs of National Income Committee, the Planning Commission and various ministries of the government. Originally under the Ministry of Finance, Directorate of NSS was transferred to Cabinet Secretariat in 1957. In 1969 the Directorate was turned into the National Sample Survey Organization which comes under the Department of Statistics Ministry of Planning. National Sample

Survey is being conducted since its inception in the form of its successive rounds.

THE MAIN FUNCTIONS OF THIS ORGANIZATION ARE

(i) To provide statistical and other information needed for the efficient conduct of government business.

(ii) To evolve statistical technique to bear on the analysis of information, the solution of administrative problems and the estimation of future trends.

(iii) To collect and publish information which will be of use to those engaged in economic activities in the country.

(iv) To provide and analyse information which are useful to the research workers.

(v) To assist in keeping the public informed of the new developments in the economic and the social fields.

In short, NSSO has unified control of the governing council with regard to survey designs, field operations, data processing, economic analysis and publication of NSS data. Present structure of NSSO consists of four functional divisions, with a Chief Executive Officer at the apex.

THE FOUR DIVISIONS ARE:

(a) Survey Design and Research. (b) Field Operations.

(c) Data Processing. (d) Economic Analysis.

THE IMPORTANT PUBLICATIONS ARE

(i) The reports on various rounds of NSS

(ii) The quarterly bulletin Sarvekhshana.

2.4 DIRECTORATE OF ECONOMICS AND STATISTICS J&K GOVERNMENT, ITS FUNCTIONS

INTRODUCTION:-The Directorate of Economics & Statistics has grown out of the nucleus of a "Statistical Section" in the Planning Department. The first expansion took place in 1957 and designating the unit as "

FUNCTIONS:- Collection, compilation and analysis of data emanating from day-to-day working in various Departments, conducting of sample surveys, evaluation studies, monitoring of projects and programmes are Statistical Bureau". The Directorate of Economics & Statistics, as it is now, was set up in 1967-68 and the Organisation has grown both vertically and horizontally. The Directorate of Economics & Statistics has two Regional Joint Directorates. At district level, the District Statistics & Evaluation Officers and Chief Planning Officers of the rank of Deputy Directors are functional and at the block level there is a mini Statistical Unit consisting of one Statistical Officer, one Junior Statistical Assistant and one Junior Assistant. The Directorate has Statistical Units in all important departments in the Civil Secretariat, Heads of Departments, Provincial and District level offices and in some cases even at block/tehsil level offices planning. In addition there are two Statistical Training Schools one in each division of the State.

The Statistical Personnel are also charged with preparation/co-ordination of the district and departmental plans as well. In short, the Department is catering almost to the entire Statistical and Planning requirements of the State.

The Directorate of Economics & Statistics comprises of various divisions and each division is charged with defined functions and duties as detailed hereunder:

i. Official Statistics division

ii. State Income division

iii. Vital Statistics division

iv. Economic Analysis

v. Price collection division

vi. Co-ordination and Publications

vii. Evaluation of dev. programmes/projects

viii. Survey and National Sample Survey

ix. Computer

x. Administration

It is the apex body in the official Statistical System of the State & mandated for the following responsibilities.

1- The nodal agency for Development of the Statistical system in the State.

2- Co-ordinates the Statistical work in respect of Govt, Semi Government departments & other institutes.

3- Prepares State Income estimates & District Domestic estimates and related aggregates.

4- Organises and Conducts Periodic surveys and Censuses.

5- Conducts large scale sample surveys as a part of NSSO Surveys for creating data base needed for studying the impact of specific problems for the benefit of different population groups in diverse Socio-economic areas such as employment, consumer expenditure, housing conditions & environment, literacy levels, health, nutrition, family welfare etc.

6- Disseminates Statistical information on various aspects through a no. of Publications. The publications are distributed among Government, semi Government and other users.

PUBLICATIONS RELEASED

The Directorate of Economics and Statistics publishes Statistical Products encompassing various Socio-economic activities being carried out in Government, Semi-Government and Private Institutions for the benefit of users especially Planners, Policy makers, Academicians, Scholars etc.

SOME OF THE IMPORTANT PUBLICATIONS ARE

1. Digest of Statistics
2. J & K Economic Review: Special Issue brought out for 90th Indian Economic Association meeting held at University of Kashmir
3. Indicators of Regional Development
4. J & K in Indian Economy: Annual Price Review

This publication has received appreciation from the CSO and Labour Bureau of India Pre-Budget Economic Survey The exercise carried by the DES is first of its kind. The Report on pre-budget survey was presented by the Hon'ble Minister for Planning & Finance. The work done by the Department has been appreciated at all levels in the State Special survey conducted by DES at the behest of State Govt; and appreciated by Central Statistical Organisation.

S.No	Title of Publications	Periodicity
1	Digest of Statistics	Annual
2	J & K in Indian Economy	Annual
3	Economic Review of J & K	Annual
4	Economy Survey, J & K	Annual
5	Vital Statistics Bulletin	Annual
6	State and Domestic product	Annual
7	Hand Book of Statistics	Annual
8	Quarterly Statistical Newsletter	Quarterly
9	Socio economy Profile of J & K	Adhoc
10	Land Use Statistics	Adhoc
11	Activity Profile	Adhoc
12	District Statistical Hand Book	Annual
13	District Economic Review	Annual
14	District at a Glance	Annual
15	Constituency wise amenity directories	Annual
16	Block Statistical Hand Book	Annual
17	Regional digest of statistics	Annual

18	NSS Report	Annual
19	Integrated sample Survey report-production of live	Annual
20	BPL Survey	As per set procedure

2.5 SUMMARY AND FURTHER SUGGESTED READING

The main objectives of this lesson were to introduce the students with CSO, NSSO, its working and role in Indian official statistical system and to introduce the students with DES J&K government, its functions. The CSO was set-up by the government of India in 1951 as a part of Cabinet Secretariat having coordinating and advisory functions for statistical activities of the various central and state departments; it provides national statistics to United Nations and its specialized agencies and brings out publications. The National Sample Survey (NSS), initiated in the year 1950, is a nation-wide, large-scale, continuous survey operation conducted in the form of successive rounds. It has four Divisions namely, the Survey Design and Research Division (SDRD), Field Operations Division (FOD), Data Processing Division (DPD), and Coordination and Publication Division (CPD), The Directorate of Economics & Statistics J & K Government has grown out of the nucleus of a "Statistical Section" in the Planning Department has the functions of collection, compilation and analysis of data emanating from day-to-day working in various Departments, conducting of sample surveys, evaluation studies, monitoring of projects and programmes are Statistical Bureau". The Directorate of Economics & Statistics, as it is now, was set up in 1967-68 and the Organisation has grown both vertically and horizontally.

FURTHER SUGGESTED READING

1. Goon, Gupta and Das Gupta:Fundamentals of Statistics,World Presss
2. Economic Analysis :CSO
3. Digest of statistics:DES, J&K government

SELF ASSESSMENT QUESTIONS

1. Disscuss the role of CSO in Indian official statistical system
2. Discuss the role of DES in collecting and maintaining official statistical system

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3.1 OBJECTIVES

The following are the main objectives of this lesson:

- To introduce the subject matter of this course material,
- To define the meaning of quality,
- To provide the dimensions of quality,
- To provide the terminology of quality engineering
- To provide the statistical methods for quality control and improvement.

3.2 INTRODUCTION TO QUALITY

Quality: We may define quality in many ways most of the people have a conceptual understanding of quality is relating to one or more desirable characteristics that a product or service should possess .The definition of quality in more precise and useful way is based upon the fact that the product and services must meet the requirements of those who use them thus quality means fitness for use .There are two general aspects of fitness for use, quality of design and quality of conformance.

Quality of conformance is how well the product conforms to the specifications required by the design and it is influenced by a numbers of factors such as choice of manufacturing process, the training and supervision of workforce, the type of quality used etc. This definition of quality is more associated with the conformance aspect of quality than design.

Thus we prefer the modern definition of quality which is ''Quality is inversely proportional to variability''. It means variability increases quality decreases.

“Without quality control you as a producer or purchaser, are in the same position as the man who bets on a horserace, with one exception that odds are not posted”.—F.M.Steadman

The statistical quality control is only diagnostic. It can only indicate whether the standard is being maintained. The remedial action rests with the technician. It is therefore remarked “Quality control is achieved most efficiently, of course, not by the inspection operation itself, but by getting at causes”. —Dodge and Roming.

3.3 MEANING AND DIMENSIONS OF QUALITY

MEANING:

Statistical Quality Control (SQC) refers to the statistical techniques employed for the maintenance of uniform quality in a continuous flow of manufactured products”.

“SQC is a simple statistical method for determining the extent to which quality goals are being met without necessarily checking every item produced and for indicating whether or not the variations which occur are exceeding normal expectations. SQC enables us to decide whether to reject or accept a particular product” —Grant

“SQC, is an effective system for co-ordinating the quality maintenance and quality improvement efforts of the various groups in an organisation so as to enable production at the most economical levels which allow for a full customer satisfaction” A. V. Feigenbaum

ORIGIN:

The origin of statistical quality control is only recent. It was introduced after the First World War by Walter A. Shewhart and Harold F. Dodge of the Bell Laboratories (U.S.A). They used probability theory to develop methods for predicting the quality of the products by conducting tests of the quality on samples of products turned out from the factory. During the Second World War these methods were used for testing war equipment.

Presently methods of statistical quality control are used widely in production, storage, packing transportation, etc. The tests being confined to only a part of the whole lot and at times only at suitable intervals. These methods have saved lot of time and expenditure otherwise involved in full inspection. Especially, when tests involve the destruction of the product as in the case of the test of the breaking of the glass, tensile strength of the metal, the resistance of the wire, or the working life of a bulb. A more thorough test is possible when lesser number if items are subjected to inspection and test.

SQC is thus, planned collections and effective use of data for studying causes of variations in quality either as between processes, procedures, materials, machines, etc,. or over periods of times. This cause-effect analysis is then fed back into the system with a view to continuous action on the process of handling, manufacturing, packaging, transporting and delivery for end-use.

DIMENSION OR COMPONENTS OF QUALITY:

The quality standards are normally set by the makers of the product. The quality consciousness amongst producers is always more when there is competition from rival producers. Also, when consumers are quality conscious, the continuing patronage of customers depends a great deal on maintenance of quality standards.

Quality has different connotations- in health and hospitality it may mean ‘hygiene’; in electrical and electronics, it may mean, ‘safety;’ in services it may mean ‘speed’ and ‘reliability’ and so on. In the present context even price is a quality measure! Operationally, however, quality refers to conformance to established standards.

The most fundamental definition of a quality product is one that meets the expectations of the customer. However, even this definition is too high level to be considered adequate.

In order to develop a more complete definition of quality, we must consider some of the key dimensions of a quality product or service which are as given below.

1. **Reliability**: Complex products such as automobiles, aeroplanes, will usually require some repair over their life span; if a product requires frequent repairs we say the product is unreliable. Will the product consistently perform within specifications?

Reliability may be closely related to performance. For instance, a product specification may define parameters for up-time, or acceptable failure rates. Reliability is a major contributor to brand or company image, and is considered a fundamental dimension of quality by most end-users.

2. **Performance:** Potential customer usually evaluates a product to determine if it will perform certain specific functions and determine hoe well it perform.

Does the product or service do what it is supposed to do, within its defined tolerances? Performance is often a source of contention between customers and suppliers, particularly when deliverables are not adequately defined within specifications.

The performance of a product often influences profitability or reputation of the end-user. As such, many contracts or specifications include damages related to inadequate performance.

3. **Durability:** It is effective life span of a product. Customers obviously want products that perform satisfactorily over a long period of time. How long will the product perform or last, and under what conditions?

Durability is closely related to warranty. Requirements for product durability are often included within procurement contracts and specifications. For instance, fighter aircraft procured to operate from aircraft carriers include design criteria intended to improve their durability in the demanding naval environment

4. **Serivicability:** It means how easy it is to repair the product. Thus the customer's view is directly influenced by how quickly and economically a repair activity can be performed. As end users become more focused on Total Cost of Ownership than simple procurement costs, serviceability (as well as reliability) is becoming an increasingly important dimension of quality and criteria for product selection.

5. **Preceived quality:** It means the reputation of the product or company. Perception is reality. The product or service may possess adequate or even superior dimensions of quality, but still fall victim to negative customer or public perceptions.As an example, a high quality product may get the reputation for being low quality based on poor service by installation or field technicians. If the product is not installed or maintained properly, and fails as a result, the failure is often associated with the product's quality rather than the quality of the service it receives.

6. **Conformance of standards**

It means the product made exactly as the designer intended. Does the product or service conform to the specification?

If it's developed based on a performance specification, does it perform as specified?

If it's developed based on a design specification, does it possess all of the features defined?

7 Features

Does the product or services possess all of the features specified, or required for its intended purpose? While this dimension may seem obvious, performance specifications rarely define the features required in a product. Thus, it's important that suppliers designing product or services from performance specifications are familiar with its intended uses, and maintain close relationships with the end-users

8 Aesthetics

The way a product looks is important to end-users. The aesthetic properties of a product contribute to a company's or brand's identity. Faults or defects in a product that diminish its aesthetic properties, even those that do not reduce or alter other dimensions of quality, are often causing for rejection.

3.4 QUALITY ENGINEERING TERMINOLOGY

Visual gauging, count and measurement

Inspection if raw material, semi finished and finished products is an important part of quality assurance. One may apply

(1) Inspection of each and every unit being produced i.e., 100% inspection

(2) Control charts or through sampling techniques.

In order to specify an item as good or bad, accepted or rejected a number of techniques such as visual gauging, count and measurement etc, can be applied.;

- Visual gauging technique makes use of visual inspection of the item being produced or produced. Here the items or lots are visually checked by the quality supervisor and declared as good or bad on the basis of visual inspection only. Here the decision about6 the product/ lot is taken on the basis of visual inspection only and no physical measure (measurements such height, weight, diameter, circumference etc) or some better quality assurance technique is involved. So, it doesn't insure better quality as compared with the other quality assurance measures. An item declared fit by visual gauging may turn otherwise when some other technique is applied.

- Count In some situations it is possible for a product to contain nonconformities but still it conforms to the standards or still it is declared fit. In such situations count can also used to specify an item or lot on the basis of counting number of nonconformities in each item or sample and declaring them it as of acceptable quality or some corrective measure is required. For example a manufacture of match box may count the number of match sticks in a match box thus may check the variability if any as claimed. A computer manufacturer may apply count to check the no. of defects in the printed circuit board and on the basis of sample of such boards by using statistical techniques such as control charts for attributes he may specify his computer's quality and take corrective action if required.

No doubt count and visual gauging ensures quality but here physical aspect (dimensional variability such as height, weight, diameter, circumference melting temperature etc) is not properly addressed. In order to ensure the dimensional consistency of items or lot, we apply measurement the items or lot being produced or at finished stage. This ensures the check on dimensional variability. For example a manufacturer screw may apply measurement technique to contain the variability of dimension of the screw such as diameter of the screw and whenever these dimensions falls outside the control limits he may take some corrective action so that the number of defective items may not be too excessive.

3.5 IMPORTANCE OF STATISTICAL METHODS IN INDUSTRIAL RESEARCH AND PRACTICE

In this competitive world, the success of a manufacturer mostly depends on quality of his product. Quality is becoming a key decision factor in almost all products and services. This phenomena is widespread irrespective whether the consumer is an individual, an industrial corporation or a retail store. Consequently quality is key leading to success in any industry. A manufacturer can not afford the rejection of his finished product so often. Therefore, it is necessary to keep constant vigil on the quality of the finished product

Keeping in view the above mentioned facts a lot of research has been done and advancement made in the field of quality assurance in almost all industrial sectors. Statistical methods are playing a vital role in this regard and working not only to ensure best quality to the consumers but also helpful for the producers in the form of increased productivity and decreased rework and wastage.

These objectives can be fulfilled by the statistical technique named statistical quality control (S.Q.C.). With the aid of statistical methods we can find out whether the variation in the articles is of a random nature, or whether some reason can be assigned for this variation. Thus, statistical quality control methods are applied to two phase of the manufacturing process.

(i) Control is to be maintained during the process of manufacturing of the articles. This facilitates the manufacturer to keep a constant vigil during the process. At this stage, it is statistically tested whether the variation occurring in various pieces is by chance or due to some defect in the manufacturing process like some fault in the machine. S.Q.C. is applicable to any repetitive manufacturing process and is known as a process control. The process control, which is carried through the inspection of samples collected at regular intervals, guards the producer against the production of poor quality product, and assists to determine whether the manufacturing process is running under control or not.

(ii) The second phase is the checking of the quality of the manufactured product in respect of its acceptability. This is achieved through an acceptance inspection or a sampling inspection plan. The purpose of this is to test whether the product, which is in existence, is acceptable to the consumer or not. Such a sampling inspection is often termed as product control or lot control. Product control is carried through the inspection of a sample of items, selected randomly from the lot under consideration.

In addition to these techniques there are a number of statistical tools which are helpful in analyzing quality problems and overcoming them. For instance designed experiments are extremely helpful in discovering the key variables influencing the quality characteristics in the process. In fact they are the best off line tools used during the development activities and at early stages of manufacturing.

The advantages of statistical quality control in Industrial Research and Practice are summarised below.

1. Statistical quality control makes it possible to discriminate whether the deviation from the standard occurring in the product, during manufacturing process, is due to chance factors or due to assignable causes.

2. The items, which meet the specifications under statistical control, get a good

market for sale and the demand increases day by day.

3. Quality control indicates the defects, if any in the machinery or the inefficiency of the operator

4. Statistical quality control is extremely helpful, particularly in the case, where the units are destroyed under inspection e.g., the life of an electric bulb, explosiveness of crackers, bullets or bombs, life of a battery cell etc.

5. The greatest advantage is the low cost of inspection and the assurance for the product to be of standard quality.

6. It minimizes the risk of the consumer as well as the producer.

7. Quality control techniques provide protection to the manufacturer against losses, due to the rejection of manufactured products, likely to be made at a late stage.

8. Further, an industrial establishment can also check the quality of the raw material before its consumption, through quality control scheme. This protects the producer from further losses.

9 An objective check is maintained on the quality of the product. Once the plan is chalked out it can be used by lower technical staff also.

ADVANTAGES OF STATISTICAL QUALITY CONTROL

There are two options before a manufacturer. He should either get each and every item checked and decide about the quality or he should use the statistical quality control methods, S.Q.C. involves the inspection of a small number of items and decides about the quality of the whole lot of the product. S.Q.C. has many advantages over 100 per cent inspection which are recorded below.

1. S Q C involves inspection of only a fraction of items produced in a fixed period. Hence, it is very economical.

2. The inspection of each and every item has hardly been feasible, as the rate of production in many cases will be faster than the time required for the inspection of items. Hence, 100 per cent inspection would cost too much. Also, in cases where the unit is destroyed during inspection, 100 per cent inspection is impossible.

3. The inspection of each and every unit will reduce the efficiency of the quality inspectors because of boredom. S.Q.C. keeps the quality control personnel alert.

4. S.Q.C. can be carried through persons who do not possess a high degree in engineering or statistics. As a matter of fact, great skill and intelligence is required to develop the statistical method for quality control in a particular case rather than applying the methods set for the purpose.

5. S.Q.C. keeps consistent vigilance on the quality of the product. The moment it is found that the process is out of control, the production engineer is informed about it. In this, way, there is an incalculable reduction in losses.

6. Process control provides the basis to the producer, for deciding about the specifications. It makes no sense to fix up the specifications, which can not be maintained economically.

7. S.Q.C. enables the manufacturers to know whether the changes brought in the production by installing new machines or by changing the system of the process or by employing more skilled persons has improved the quality of the product or not.

8. S.Q.C. is a basis for compromise between the machine operators and engineers. The engineers may expect a total adherence to specifications whereas the operators may emphasise their performance up to the mark, in spite of large variability in the units. Hence, S.Q.C. is a good device to keep both the sections satisfied.

9. S.Q.C. provides protection against losses to the producer as well as to the consumer.

SUMMARY AND FURTHER SUGGESTED READING

It should be obvious from the discussion above that the individual dimensions of quality are not necessarily distinct. Depending on the industry, situation, and type of contract or specification several or all of the above dimensions may be interdependent.

When designing, developing or manufacturing a product (or delivering a service) the interactions between the dimensions of quality must be understood and taken into account.

While these dimensions may not constitute a complete list of relevant dimensions, taking them into consideration should provide us with a better understanding of the slippery concept of quality.

FURTHER SUGGESTED READING

1. D.C Montgomery. An Introduction to Statistical Quality Control.
2. Goon, Gupta and Das Gupta:Fundamentals of Statistics,World Presss

SELF ASSESSMENT QUESTIONS:

1. Define inspection in reference to SQC and give its type.
2. Define tolerance limits and specification limits. Also give the types of tolerance limits?
3. Describe the tools used for statistical quality control.

4.1 OBJECTIVES

The main objectives of this lesson are :-

- To introduce the students with quality factors
- To introduce the students with various aspects of quality
- determination of natural tolerance limits

4.2 MEASUREMENT AND DETERMINATION OF QUALITY FACTORS

Defines the customer's expectations for quality, the internal process and product attributes that indicate whether the quality factors are being satisfied, and the measures to be used to give visibility to the levels of quality being achieved.

The quality requirements are defined in terms of quality factors, quality criteria and quality metrics.

Quality Factors: The quality factors document the user-perceived aspects of the end products that will determine whether the product meets the customer's expectations.

Quality Criteria: The quality criteria document the internal process and product attributes that will be monitored throughout the project to indicate whether the quality factors are being satisfied.

Quality Metrics: The quality metrics document the indicators to be used to measure service and product quality.

Count and measurement and visual gauging

Inspection if raw material, semi finished and finished products is an important part of quality assurance. One may apply

(i) Inspection of each and every unit being produced i.e., 100% inspection

(ii) Control charts or through sampling techniques.

In order to specify an item as good or bad, accepted or rejected a number of techniques such as visual gauging, count and measurement etc, can be applied.;

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4.3 IMPORTANCE OF STATISTICAL METHODS IN INDUSTRIAL RESEARCH AND PRACTICE

In this competitive world, the success of a manufacturer mostly depends on quality of his product. Quality is becoming a key decision factor in almost all products and services. This phenomena is widespread irrespective whether the consumer is an individual, an industrial corporation or a retail store. Consequently quality is key leading to success in any industry. A manufacturer can not afford the rejection of his finished product so often. Therefore, it is necessary to keep constant vigil on the quality of the finished product

Keeping in view the above mentioned facts a lot of research has been done and advancement made in the field of quality assurance in almost all industrial sectors. Statistical methods are playing a vital role in this regard and working not only to ensure best quality to the consumers but also helpful for the producers in the form of increased productivity and decreased rework and wastage.

These objectives can be fulfilled by the statistical technique named statistical quality control (S.Q.C.). With the aid of statistical methods we can find out whether the variation in the articles is of a random nature, or whether some reason can be assigned for this variation. Thus, statistical quality control methods are applied to two phase of the manufacturing process.

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(ii) The second phase is the checking of the quality of the manufactured product in respect of its acceptability. This is achieved through an acceptance inspection or a sampling inspection plan. The purpose of this is to test whether the product, which is in existence, is acceptable to the consumer or not. Such a sampling inspection is often termed as product control or lot control. Product control is carried through the inspection of a sample of items, selected randomly from the lot under consideration.

In addition to these techniques there are a number of statistical tools which are helpful in analyzing quality problems and overcoming them. For instance designed experiments are extremely helpful in discovering the key variables influencing the quality characteristics in the process. In fact they are the best off line tools used during the development activities and at early stages of manufacturing.

4.4 TYPES OF INSPECTION

The different options available to buyers, when it comes to the representativity of inspection findings.

Inspection level II (under "normal severity") is appropriate for most inspections. But it is sometimes necessary to increase or reduce the number of samples to check.

Why different inspection levels?

There is a fairly obvious principle in statistical quality control: the greater the order quantity, the higher the number of samples to check.

But should the number of samples only depend on the order quantity? What if this factory had many quality problems recently and one suspect there are many defects? In this case, one might want more products to be checked.

On the other hand, if an inspection requires tests that end up in product destruction, shouldn't the sample size be drastically reduced? And if the quality issues are always present on all the products of a given batch (for reasons inherent to processes at work), why not check only a few samples?

For these reasons, different levels are proposed by MIL-STD 105 E (the widely recognized standard for statistical quality control).

It is usually the buyer's responsibility to choose the inspection level–more samples to check means more chances to reject bad products when they are bad, but it also means more days (and amount) spent in inspection.

The Three "general" inspection levels are

Level I: Has this supplier passed most previous inspections? Do you feel confident in their products quality? Instead of doing no quality control, buyers can check less samples by opting for a level-I inspection.

However, settling on this level by default, in order to spend less time/money on inspections, is very risky. The likelihood of finding quality problems is lower than generally recommended.

Level II: It is the most widely used inspection level, to be used by default.

Level III: If a supplier recently had quality problems, this level is appropriate. More samples are inspected, and a batch of products will (most probably) be rejected if it is below the quality criteria defined by the buyer.

Some buyers opt for level-III inspections for high-value products. It can also be interesting for small quantities, where the inspection would take only one day whatever the level chosen.

The 4 "special" inspection levels: These special levels can be applied in cases where only very few samples can be checked. "Four additional special levels, S-1, S-2, S-3 and S-4 […] may be used where relatively small sample sizes are necessary and larger sampling risks can be tolerated" (ISO 2859 standard).

Under S-3 level, the number of samples to check is lower than under S-4, and so on.

In practice for consumer goods, quality control is usually performed under the general levels. The special levels are used only for certain tests that either take lots of time or destroy the samples. Another situation where special levels are appropriate is a container-loading supervision–to have an idea of what is inside the cartons, without spending too much time at that checking.

In particular the choice among various types of inspection procedure is essentially an economic one in making the decision regarding acceptance inspection for a particular purpose; it may be desirable to consider not only various procedures of acceptance but also the alternatives of

(1) No inspection at all (2) 100% inspection

(3) Possibility of acceptance sampling by variables.

However an important element in selection of inspection procedure should be the probable contribution of the procedure to quality improvement.

Illustration: In general there are there types of inspection procedures which are employed as per requirement of the quality.

(1) **Normal Inspection:** Normal, inspection is used at the start of inspection activity either in single sampling, double sampling or multiple sampling. This type of inspection procedure may be studied in tabular manner as given below while considering N=2000

Sample Size	Acceptance Number	Rejection Number
80	0	3
80	3	4

Thus from, a lot of 2000, inspect a random sample of n1=80 units if there are no defective, accept the lot. If there are three or more defectives reject the lot if there are 1 0r 2 defectives take a second sample of size n2=80. If the combined number of defectives is 3 or less accept the lot, if there are 4 or more defectives then reject the lot.

(2) **Tightened Inspection:** This inspection activity is initiated when the vendor's recent quality history has deteriorated. Acceptance requirement under tightened inspection are more stringent than under normal inspection. It is remarkable that under normal inspection, tightened inspection is initiated when 2 out of 5 consecutive lots have been rejected on original inspection. This inspection procedure may be studied in tabular manner as given below

Sample size	Acceptance Number	Rejection Number
80	0	2

Thus from a lot a random sample of n1=80 units is drawn if there are no defective, accept the lot. If there are two or more defectives reject the lot if there is 1 defective take a second sample of size n2=80. If the combined number of defectives is 1 accept the lot, otherwise reject the lot.

(3) **Reduced Inspection:** This inspection activity is initiated when the vendor's recent quality history is exceptionally good. The reduced inspection is initiated if the following four conditions are satisfied

1. The preceding 10 lots have been on normal inspection and none of them has been rejected.
2. Total number of defectives in preceding 10 lots is less than or equal to applicable limit number.
3. Production is at steady rate i.e., there is no difficulty such as machine breakdowns, material; shortage etc.
4. Reduced inspection is considered desirable by the authority responsible for sampling. It may be described in a tabular manner as given below.

Sample size	Acceptance Number	Rejection Number
32	0	3
32	0	4

Take a random sample of n1=32 units from the lot, if there are no defective, accept the lot. If there are three or more defectives reject the lot if there is 1 or 2 defective take a second sample of size n2=32 units. If the combined number of defectives is 1,2 or 3 accept the lot, but return to the normal inspection.

Two examples to get an clearer understanding

Let's say you have ordered 5,000 pcs of a product. In the table below, you can see how many samples would be drawn under each of the 7 inspection levels.

General inspection levels			Special inspection levels			
I	II	III	S-1	S-2	S-3	S-4
80pcs	200pcs	315pcs	5pcs	8pcs	20pcs	32pcs

As you can see, the numbers of samples to check vary from 5pcs to 315pcs. But a trained inspector might be able to do it in one day, whatever the inspection level you choose.

Now let's say you have ordered 40,000pcs of a product. Again, you can see the differences in sample sizes.

General inspection levels			Special inspection levels			
I	II	III	S-1	S-2	S-3	S-4
200pcs	500pcs	800pcs	8pcs	13pcs	32pcs	80pcs

In this case, the inspection might take one day of work (for S-1, S-2, S-3, S-4, or reduced level), two days (under level II), or three days (under level III).

4.4 DETERMINATION OF NATURAL TOLERANCE LIMITS

Tolerance Limits: The control charts may show that the process is in control at a particular level. But it may also be of interest to know whether the process meet the specification limits set for the items, a decision on this regard may be made by comparing what we say as natural tolerance limits of the process with the specification limits. If μ and σ are the process average and standard deviation respectively then the limits $\mu \pm 3\sigma$ will be called as the natural tolerances of the process.

If the estimated tolerance limits are not included within the specification limits then a readjustment of the process will be advisable. If the estimated tolerance limits lie within the specification limits, this will signify that process is too good.

The ideal situation will be attained when tolerance limits are approximately coincident withy the specification limits.

These limits may be categorised into two classes depending upon the form of the distribution and its parameters are known or not.

Tolerance limits based on normal distribution

Non-Parametric Tolerance Limits (Distribution free Tolerance limits)

1. **TOLERANCE LIMITS BASED ON NORMAL DISTRIBUTION**:

2. Suppose a random variable X is normally distributed with mean μ and variance σ^2, both are unknown. Now choose random sample of n observations. We can calculate the sample mean $\overline{X}$ and sample variance s^2.Usually tolerance limits are

$\mu \pm Z_{\alpha/2}\sigma$,but here μ and variance σ^2 are unknown and their estimated values can be used to obtained tolerance limits viz.

$$\mu \pm Z_{\alpha/2} s$$

Since and are only estimates and are not true parameter values, we can't say that the above interval always contain of the distribution. However, a constant k may be determined so that will include at least of the distribution. The value of k for can be obtained from the tables which are readily available.

For two sided tolerance limits the no of observations that must be taken to ensure that with the probability υ at least $100(1-\alpha)\%$ of the distribution will lie between largest and smallest observation obtained in the sample is

$$n \cong \frac{1}{2} + \left(\frac{2-\alpha}{\alpha}\right)\frac{\chi^2_{1-u,4}}{4} \text{ approximately.}$$

e.g., 99% certain that at least 95% of the population will be included between the sample extreme values is given by

$$n \cong \frac{1}{2} + \left(\frac{2-0.05}{0.05}\right)\frac{\chi^2_{1-0.99,4}}{4} = n \cong \frac{1}{2} + \left(\frac{1.95}{0.05}\right)\frac{13.28}{4} = 130$$

It is noticeable that the basis difference between control limits and tolerance limits is that control limits are used to provide an interval estimate of the parameter where as tolerance limits are used to indicate the limits between which we can expect to find a specified proportion of the population

Example: a manufacturer of solid fuel rocket propellant is interested in finding tolerance limits of the process such that 95% of the burning rates will lie within these limits with probability 0.99. It is known from the previous experience that burning rate is normally distributed. A random sample of 25 observations shows that the sample mean and variance are $\overline{X}$ =40.75 and s^2=1.87

Sol: Here we are given n=25 , $\overline{X}$ =40.75 , s^2=1.87 $\alpha = 0.05$ $\nu = 0.99$

Tolerance Limits are given by

$\mu \pm ks = 40.75 \pm k(1.037)$

Here we find k from the table corresponding to given n, α and ν which is 2.972

Hence the tolerance limits are

$40.75 \pm 2.972(1.037) = 40.75 \pm 4.06 = 36.69, 44.81$

4.6 SUMMARY AND FURTHER SUGGESTED READING

The main objectives of this lesson were to introduce the students with factors wish influence the quality and to introduce the students with various aspects of quality and how the inspection activity takes place, how tolerance limits are determine.

Inspection if raw material, semi finished and finished products is an important part of quality assurance. It can be achieved through 100% inspection, Control charts or through sampling techniques.

Quality is becoming a key decision factor in almost all products and services. It is key leading to success in any industry. A manufacturer can not afford the rejection of his finished product so often. Therefore, it is necessary to keep constant vigil on the quality of the finished product. The control charts may show that the process is in control at a particular level. But it may also be of interest to know whether the process meet the specification limits set for the items, a decision on this regard may be made by comparing what we say as natural tolerance limits of the process with the specification limits **.**

FURTHER SUGGESTED READING

1. D.C Montgomery. An Introduction to Statistical Quality Control.
2. Goon, Gupta and Das Gupta:Fundamentals of Statistics,World Presss

SELF ASSESSMENT QUESTIONS

1. Define inspection in reference to SQC and give its type.
2. Define tolerance limits and specification limits. Also give the types of tolerance limits?
3. Describe the tools used for statistical quality control.

UNIT - II

Lesson - 5

Lesson-6

Lesson-7

5.1 OBJECTIVES

The following are the main objectives of this lesson:

- To provide the general theory of statistical quality control,
- To explain the causes of variation in quality of product,
- To introduce the concept of Statistical Quality Control,
- To explain the different control limits,
- To provide the concept of sub-grouping.

5.2 CASUSES OF VARIATION IN QUALITY

The basis of statistical quality control is the degree of '**variability**' in the size or the magnitude of a given characteristic of the product. Variation in the quality of manufactured product in the repetitive process in industry is inherent and inevitable. Some amount of variability is bound to be there, however, scientific and accurate the production process is. The various causes of variation may be classified into Chance and Assignable Causes of Variation:

(ii) Chance causes, and

(i) Assignable causes.

Chance Causes. These causes have nothing to do with any latent or patent defect in the production process. These arise in the process of taking out samples and drawing inferences. It is difficult to assign any specific cause for these variations.

The purpose of statistical quality control as stated by **Duncan** is to separate these assignable causes from the chance or random causes. Here we are more interested in the variations within the sample and not between samples. The latter are there irrespective of the fact the lots are drawn during a continuous process. The statistical process of achieving this objective is to lay down the limits of chance variations (including those between sample variations) any variations beyond those limits must be due to assignable causes within samples. If it is found that the process is out of control the specific causes may be looked into through technical examination of the production process in its various stages. It may be noted that statistical methods may also be necessary to detect if each process of the whole production process is within control. Some **"stable pattern of variation" or "a constant cause system"** is inherent in any particular scheme of production and inspection. This pattern results from many minor causes that behave in a random manner. The variation due to these causes is beyond the control of human hand and cannot be prevented or eliminated under any circumstances. One has got to allow for variation within this stable pattern, usually termed as allowable variation. In such a situation, the process is said to be under statistical control.

Assignable causes: If the articles show marked deviation from the given specifications of a product, the utility of articles is in jeopardy. In that situation, one has to make a search for the causes responsible for the large variation in the product. The causes due to faulty process and procedure are known as assignable causes. The variation due to assignable causes is of non-random nature. The assignable causes may creep in at any stage of the process, right from the arrival of the raw materials to the final delivery of goods. Some of the important factors of assignable causes of variation are substandard or defective raw material, new techniques or operations, negligence of the operators, wrong or improper handling of machines, faulty equipment, unskilled or inexperienced technical staff and so on. These causes can be identified and eliminated and are to be discovered in a production process before it goes wrong i.e., before the production becomes defective.

The main purpose of Statistical Quality Control (S.Q.C.) is to devise statistical techniques which would help us in separating the assignable causes from the chance causes, thus enabling us to take immediate remedial action whenever assignable causes are present. the, elimination of assignable causes of erratic fluctuations is described as bringing a process under control.

A production process is said to be in a state of statistical Control, if it is governed by chance causes alone, in the absence of assignable causes of variation.

Uses of S.Q.C. We outline below briefly some of the advantages that might result when a process is brought in good statistical control.

1. The act of getting a process in statistical control helps in the detection and correction of many production troubles and brings about a substantial improvement in the product quality and reduction of spoilage and rework.

2. It tells us when to leave a process alone and when to take action to correct troubles, thus preventing frequent and unwarranted adjustments.

3. If testing is destructive (e.g., testing the breaking strength of chalk; proofing of ammunition, explosives, crackers, etc.), a process in control gives confidence in the quality of untested product which is not the case otherwise.

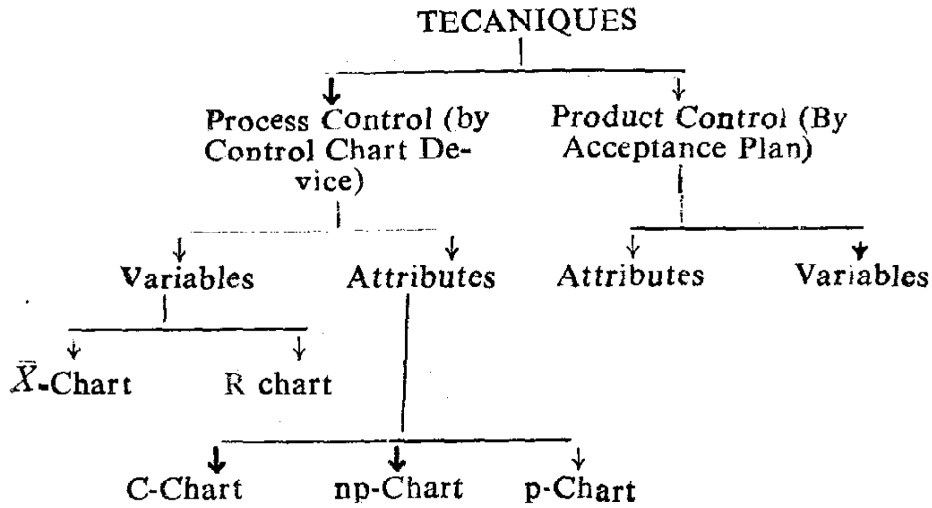
4. It provides better quality assurance at lower inspection cost.

5. Quality control finds its applications not only in the sphere of production, but also in other areas like packaging, scrap and spoilage, recoveries, advertising, etc. Foreign trade items of developing countries like India are particularly appropriate for every type of quality control in every possible area.

6. S.Q.C. reduces waste of time and material to the absolute minimum by giving an early warning about the occurrence of defects. Savings in terms of the factors stated above means less cost of production and hence may ultimately lead to more profits.

   In short, S.Q.C is a productivity enhancing and regulatory technique with three factors—Management, Methods and Mathematics.

5.3 PRODUCT AND PROCESS CONTROL



There are two broad ways of statistical control of the quality of a product, viz., process control and the product control

(a) **Process Control**. This is concerned with controlling the quality of the goods manufactured in the process of production. Process control detects whether the production process is going on in the desired fashion. In other words it controls quality of the goods to be produced. It ensures that the machines are turning out the product of a requisite standard. This is achieved largely through control chart device.

Statistical process control uses sampling and statistical methods to monitor the quality of an ongoing process such as a production operation. A graphical display referred to as a control chart provides a basis for deciding whether the variation in the output of a process is due to common causes (randomly occurring variations) or to out-of-the-ordinary assignable causes. Whenever assignable causes are identified, a decision can be made to adjust the process in order to bring the output back to acceptable quality levels.

Summing up, the main objective in any production process is to control and maintain the quality of the manufactured product so that it conforms to specified quality standards. In other words, we want to ensure that the proportion of defective items in the manufactured product is not too large. This is called ‘process control’ and is achieved through the technique of control charts pioneered by W.A. Shewhart.

(b) Product Control. This is concerned with classification of raw materials or finished goods into say acceptable, non-acceptable or whether another sample has to be tested. It is concerned with the inspection of the goods already produced, whether these are fit to be dispatched. It is necessary to note that even when the process is under control individual products may turn out to be non-acceptable. It is also not necessary that product control in the case of inputs shall ensure that the process is under control. Actually, process control is concerned more with operations, machines and hands while product control is concerned with the quality of the product turned out. Certainly a good process control will not require a strict product control.

By product control we mean controlling the quality of the product by critical examination at strategic points and this is achieved through ‘Sampling Inspection Plans’ pioneered by Dodge and Romnig. Product control aims at guaranteeing a certain quality level to the consumer regardless of what quality level is being maintained by the producer. In other words, it attempts to ensure that the product marketed by sale department does not contain a large number of defective (unsatisfactory) items.

5.4 INTRODUCTION TO CONTROL CHARTS

Control charts are the devices to describe the patterns of variation. The control charts were developed by the physicist, Dr. Walter A. Shewhart of Bell Telephone Company in 1924.

Control chart, as conceived and devised by Shewhart, is a simple pictorial device for detecting unnatural patterns of variations in data resulting from repetitive processes i.e., control charts provide criteria for detecting lack of statistical control.

Control charts can be classified by the type of data they contain. For instance, an x-chart is employed in situations where a sample mean is used to measure the quality of the output. Quantitative data such as length, weight, and temperature can be monitored with an x-chart. Process variability can be monitored using a range or R-chart. In cases in which the quality of output is measured in terms of the number of defectives or the proportion of defectives in the sample, an np-chart or a p-chart can be used.

Control charts are simple to construct and easy to interpret and tell us whether the sample point falls within 3- control limits or not. Any sample point going outside the control limits is an indication of the lack of statistica1 control, i.e., presence of some assignable causes of ‘variation which must be traced, identified and eliminated.

5.5 GENERAL THEORY OF CONTROL CHARTS

CONTROL CHARTS

The control charts are the graphic devices developed by Shewart for detecting unnatural pattern of variations in data in their respective processes. The criteria for detecting lack of control are indicated on a graph. There are three horizontal lines on the graph. These are

1. A control line to indicate the desired standard or the control level of the process
2. An upper control limits indicating the upper limit of tolerance;
3. A lower control limits indicating the lower limit of tolerance.

The control line as well as the upper and lower limits are established by computations based on the past records or current production records. However, the reliability of statistical formulae used has been proved beyond doubt in practice**.**

All control charts are constructed in a similar fashion. For example, the centre line of a chart corresponds to the mean of the process when the process is in control and producing output of acceptable quality. The vertical axis of the control chart identifies the scale of measurement for the variable of interest. The upper horizontal line of the control chart, referred to as the upper control limit, and the lower horizontal line, referred to as the lower control limit, are chosen so that when the process is in control there will be a high probability that the value of a sample mean will fall between the two control limits. Standard practice is to set the control limits at three standard deviations above and below the process mean. The process can be sampled periodically. As each sample is selected, the value of the sample mean is plotted on the control chart. If the value of a sample mean is within the control limits, the process can be continued under the assumption that the quality standards are being maintained. If the value of the sample mean is outside the control limits, an **out-of-control conclusion** points to the need for corrective action in order to return the process to acceptable quality levels.

Major Parts of a Control Chart. A control chart generally includes the following four major parts.

1. **Quality Scale.** This is a vertical scale. The scale is marked according to the quality characteristic (either in variables or in attributes) of each sample.

2. **Plotted Samples.** The qualities of individual items of a sample are not shown on a control chart. Only the quality of the entire sample represented by a single value (a statistic) is plotted. The single value plotted on the chart is in the form of a dot (sometimes a small circle or a cross).

3. **Sample Numbers.** The samples plotted on a control chart are numbered individually and consecutively on a horizontal line. The line is usually placed at the bottom of the chart. The samples are also referred to as sub-groups in statistical quality control. Generally 25 subgroups are used in constructing a control chart.

The success of control chart technique depends largely upon the efficient grouping of items into samples, such that variation in quality among items within the same sample is small, but variation between one sample and another is as large as possible. Such a sample is known as 'rational subgroup'.

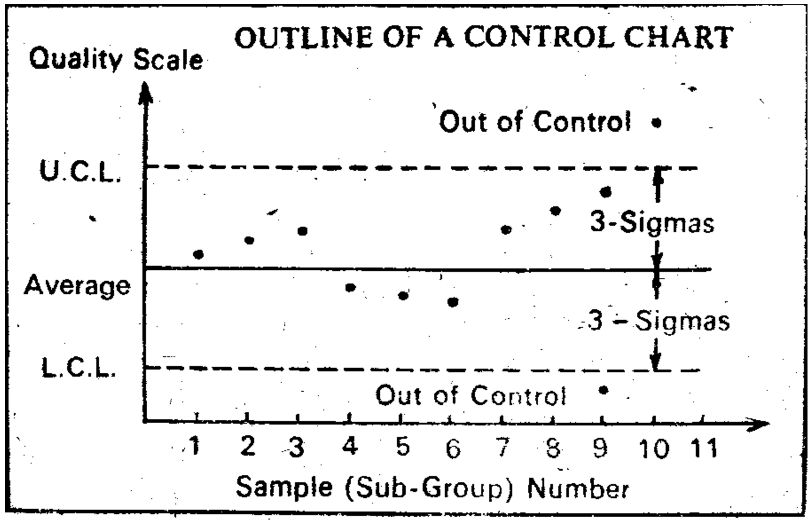
A typical control chart consists of three horizontal lines

(i) A central line (CL) (ii) Upper Control Limit (UCL)

(iii) Lower Control Limit (LCL),

If t is the underlying statistic then these values depend on the sampling distribution of t and are given by

$$UCL = E(t) + S.E(t)$$
$$LCL = E(t) - S.E(t)$$
$$CL = E(t)$$



5.6 USE OF 3-σ LIMITS IN CONTROL CHARTS

In the control chart, two kinds of errors may be made

(i) The sample point may fall outside the control limits, and consequently we may be looking for the presence of assignable causes, when really none is present.

(ii) The sample point may fall within the control limits, so that we may not be aware of the actual trouble, although assignable causes may be operating in the process.

If control limits are set at wider distance from the Central Line, say at plus and minus four times the standard deviation, the possibility of the first kind of error will diminish, and that of the second kind will increase. If control limits are closer, say at two times the standard deviation from the Central Line, the reverse will be the case. Experience has shown that control limits set at about three times the standard deviation. To provide a good economic balance between these two types of errors. As such, 3-σ control limits are commonly used unless there are special reasons for using any other limits.

If the statistic t is normally distributed with mean μ and variance σ^2 then from the fundamental area property of the normal distribution, we have

$$P[\mu_t - 3\sigma_t < \mu_t < \mu_t + 3\sigma_t] = 0.9973$$

$$\Rightarrow \qquad P\left[|t - \mu_t| > 3\sigma_t\right] = 0.0027$$

In other words, the probability that a random value of t goes outside the 3-σ limits, viz., $\mu \pm 3\sigma$ is 0 .0027, which is very small.

More closely normal, the closer the underlying population is to a normal distribution, but the principle applies in any case.

Moreover, according to the central limit theorem in probability, the statistics of observations drawn from non-normal populations will exhibit nearly normal behavior. Hence, even for non-normal population 3-σ limits are almost universally used, as they have been found to be most suitable empirically in the sense that the 3-σ control charts have been found to give excellent protection against both types of wrong actions we can take viz., looking for trouble when there is none and not looking for trouble when there really is one.

5.7 SUMMARY AND FURTHER SUGGESTED READING

Variation in the quality of manufactured product in the repetitive process in industry is inherent and inevitable. Some "stable pattern of variation" or "a constant cause system" is inherent in any particular scheme of production and inspection. This pattern results from many minor causes that behave in a random manner. The variation due to these causes is beyond the control of human hand and cannot be prevented or eliminated under any circumstances. One has got to allow for variation within this stable pattern, usually termed as allowable variation. In such a situation, the process is said to be under statistical control.

The causes due to faulty process and procedure are known as assignable causes. The variation due to assignable causes is of non-random nature. It may creep in at any stage of the process, right from the arrival of the raw materials to the final delivery of goods. These causes can be identified and eliminated and are to be discovered in a production process before it goes wrong i.e., before the production becomes defective.

The main purpose of Statistical Quality Control (S.Q.C.) is to devise statistical techniques which would help us in separating the assignable causes from the chance causes, thus enabling us to take immediate remedial action whenever assignable causes are present.

To achieve this we make use of statistical process control or product control. Statistical process control uses sampling and statistical methods to monitor the quality of an ongoing process such as a production operation Control charts)

By product control we mean controlling the quality of the product by critical examination at strategic points and this is achieved through 'Sampling Inspection Plans' pioneered by Dodge and Roming.

FURTHER SUGGESTED READINGS

1. E.L. Grant and R.L. Leavenworth. Statistical Quality Control. Tata McGraw Hill Publishing Company Limited, New Delhi.

2. D.C. Montgomery: Introduction to Statistical Quality Control. John Wiley and Sons,

5.8 SELF ASSESSMENTS

1. What do you understand by control charts and control limits? Explain.

2. Explain the concept and utility of limits in statistical quality control.

3. Describe the various tools for statistical quality control.

4. Explain the fundamentals and advantages of control charts.

6.1 OBJECTIVES

The following are the main objectives of this lesson:

- To provide the knowledge about sub grouping in control charts.
- To explain the types of control charts.
- To introduce the concept of Statistical Quality Control,
- To provide the concept of sub-grouping.

6.2 SUB GROUPING IN CONTROL CHARTS

A fundamental idea in the use of control charts is to collect sample data according to what Shewhart called the **rational subgroup** concept. Generally, this means that subgroups or samples should be selected so that to the extent possible, the variability of the observations with in a subgroup should include all the chance or natural variability and exclude the assignable variability. Then, the control limits will represent bounds for all the chance variability and not the assignable variability. Consequently, assignable causes will tend to generate points that are outside of the control limits, while chance variability will tend to generate points that are within the control limits. When control charts are applied to production processes, the time order of production is a logical basis for rational subgrouping. Even though time order is preserved, it is still possible to form subgroups erroneously. If some of the observations in the subgroup are taken at the end of one 8-hour shift and the remaining observations are taken at the start of the next 8-hour shift any differences between shifts might not be detected. Time order is frequently a good basis for forming subgroups because it allows us to detect assignable causes that occur over time. Two general approaches to constructing rational subgroups are used. In the first approach, each subgroup consists of units that were produced at the same time (or as closely together as possible). This approach is used when the primary purpose of the control chart is to detect process shifts. It minimizes variability due to assignable causes within a sample, and it maximizes variability between samples if assignable causes are present. It also provides better estimates of the standard deviation of the process in the case of variables control charts. This approach to rational subgrouping essentially gives a “snapshot” of the process at each point in time where a sample is collected.

In the second approach, each sample consists of units of product that are representative of all units that have been produced since the last sample was taken. Essentially, each subgroup is a random sample of all process output over the sampling interval. This method of rational subgrouping is often used when the control chart is employed to make decisions about the acceptance of all units of product that have been produced since the last sample. In fact, if the process shifts to an out-of-control state and then back in control again between samples, it is sometimes argued that the first method of rational subgrouping defined above will be ineffective against these types of shifts, and so the second method must be used. When the rational subgroup is a random sample of all units produced over the sampling interval, considerable care must be taken in interpreting the control charts. If the process mean drifts between several levels during the interval between samples, the range of observations within the sample may consequently be relatively large. It is the within-sample variability that determines the width of the control limits on an chart, so this practice will result in wider limits on the chart. This makes it harder to detect shifts in the mean. In fact, we can often make *any* process appear to be in statistical control just by stretching out the interval between observations in the sample. It is also possible for shifts in the process average to cause points on a control chart for the range or standard deviation to plot out of control, even though no shift in process variability has taken place. There are other bases for forming rational subgroups. For example, suppose a process consists of several machines that pool their output into a common stream. If we sample from this common stream of output, it will be very difficult to detect whether or not some of the machines are out of control. A logical approach to rational subgrouping here is to apply control chart techniques to the output for each individual machine. Sometimes this concept needs to be applied to different heads on the same machine, different workstations, different operators, and so forth. The rational subgroup concept is very important. The proper selection of samples requires careful consideration of the process, with the objective of obtaining as much useful information as possible from the control chart analysis.

Rational subgroups are those that contain or enclose only the chance or common causes. Within any rational subgroup, the variation among the readings should be due to only the natural process variation, and there should be little or no opportunity for assignable causes to add to this variation.

Rational subgroups are those that contain or enclose only the chance or common causes. Within any rational subgroup, the variation among the readings should be due to only the natural process variation, and there should be little or no opportunity for assignable causes to add to this variation.

REASONS FOR FORMING RATIONAL SUBGROUPS:

1. The variation within the subgroups can be pooled to give a good estimate of the natural process variation.

2. The presence of special causes can easily be detected, since they are responsible for any large variations between the subgroups.

Determine the subgroup size and sampling frequency. Typically, subgroups of three, five or seven units are used, and subgroups of five are most common. Odd-sized subgroups are usually selected because no calculations are necessary when the subgroup size is odd. The subgroup size affects the sensitivity of the control chart,smaller subgroups create a control chart that is less sensitive to changes in the process, while larger subgroups may be too sensitive to small, economically unimportant changes. If possible, collect data from 20-25 subgroups, with at least 100 individual values. While the data is being collected, minimize disturbances to the process. If a process change is unavoidable, we should develop a system for recording changes so that their effect can be determined.

The success of the control chart technique depends largely upon the efficient grouping of items into samples, such that variation in quality among items within the same sample is small, but variation between one sample and another is as large as possible. Such a sample is known as 'rational sub-group'. Usually, the order of production is taken as a basis for obtaining rational sub-groups, the consecutive samples being taken from the production line at intervals of time, say about 30 minutes. If articles are produced in more than one machine, the source may also be used for rational sub-grouping.

6.3 ADVANTAGES OF CONTROL CHARTS

We outline below briefly some of the advantages that might result when a process is brought in good statistical control by using the control charts

1. A control chart tells when to leave the process alone and when to take action to correct the trouble. The elimination of assignable causes of variation is described as bringing the process under control. The act of getting a process in statistical control helps in the detection and correction of many production troubles and brings about a substantial improvement in the product quality and reduction of spoilage and rework.
2. The quality control chart is used to detect shifts in the process average from the desired level. This information can be of help to management in taking appropriate steps either to change the process itself or, to install new machinery or to alter the product specification. As a matter of fact, the chart provides an unerring method of predicting the future trend of output. . In short, it tells us when to leave a process alone and when to take action to correct troubles, thus preventing frequent and unwarranted adjustments
3. The idea of control chart can be extended to the comparison of budgetary figures with actual performance. Such comparison may either.
4. If testing is destructive (e.g., testing the breaking strength of chalk; proofing of ammunition, explosives, crackers, etc.), a process in control gives confidence in the

quality of untested product which is not the case otherwise.

5. It provides better quality assurance at lower inspection cost.

6. Quality control finds its applications not only in the sphere of production, but also in other areas like packaging, scrap and spoilage, recoveries, advertising, etc. Foreign trade items of developing countries like India are particularly appropriate for every type of quality control in every possible area.

7. Quality control enables a production process to be brought into or held in a state of statistical control, i.e., one in which variability is the result of chance causes alone. So long as statistical control continues, specifications can be accurately predicted for the future, which even 100 per cent inspection cannot guarantee. Consequently it is possible to assess whether the production process is capable of turning out products which will comply with the given set of specifications.

8. The statistical quality control technique helps in more economical inspection on the basis of samples. A 100 per cent inspection is no longer necessary. Only when the process is shown to be out of control, inspection becomes necessary.

9. With the help of charts, we can also decide about the change in the method of production and separating good products from the bad ones.

6.4 TYPES OF CONTROL CHARTS

CONTROL CHARTS FOR VARIABLES AND ATTRIBUTES

Control charts are of two Types of Control Charts. Broadly the control charts can be grouped under the following two heads:

(a) Control charts for variables, (b) Control charts for attributes.

(a) **Control charts for Variables**

Many quality characteristics of a product are measurable and can be expressed in specific-units of measurement such as diameter of a screw, tensile strength of steel pipe, specific resistance of a wire, life of an electric bulb, etc. Such variables are of continuous type, the frequency distributions of which follow normal laws. For purposes of control, two types of control charts are used—one for the mean of measurement ($\overline{X}$ -chart) and another for the range of the measurement (R-chart). Depending on the nature of quality which it is desired to control. If quality is stated in terms of a variable, e.g. diameter, tensile strength etc., we have Control Charts for Variables. Three such charts are in common use—

(a) Mean chart ($\overline{X}$ -chart),

(b) Range chart (R- chart),

(c) Standard Deviation chart (σ -chart)

(b) **Control charts for Attributes**: Sometimes the quality characteristics of a product are not amenable to measurement. Such characteristics can only be identified by their absence or presence in the product. Also the produced item may be classified as good or bad. Some theoretical distributions discussed earlier are used in case of the control charts for the attributes. For example, sometimes the quality cannot be measured directly, or it is not measured for economy, and the products are classified by an attribute. These are known as Control Charts for Attributes.

For example, very often articles are inspected or tested in batches and classified either as 'good' (i.e. those which conform to specifications) or 'defective' (i.e. those which are considered as useless rejections). The quality of a product is then measured by "fraction defective" $p = d/n$, where p is a fraction and d is the number of defectives found in a sample of n articles inspected. Again, a single product or a specified size of the product may contain one or more defects, e.g. paper, textiles, metallic sheets, pipes etc. The number of defects c is then used as a quality measure. Two important control charts for attributes are

(a) Fraction Defective Chart (p chart)

(b) Number of Defectives chart (np chart)

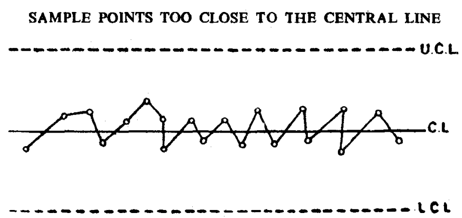
(c) Number of Defects chart (c chart)

6.5 CRITERION FOR DETECTING LACK OF CONTROL

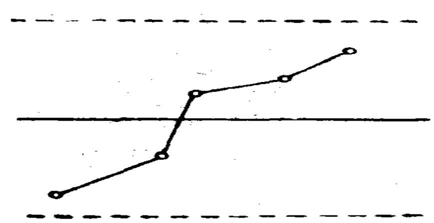
Criterion for Detecting Lack of Control in $\overline{X}$ and R charts. The main object of the control chart is to indicate when a process is not in control. The criteria for detecting lack of control are, therefore, of fundamental and crucial importance. The following situations depict lack of control

1. **A point outside the control limits'**. The probabilistic considerations provide a basis for hunting for lack of control in such a situation. A point going outside control limits is a clear indication of the presence of assignable causes of variation which must be searched and corrected.

2. **A run of seven or more points**. Although all the sample points are within control limits usually the pattern of points in the chart indicates assignable causes. One such situation is a run of 7 or more points above or below the central line in the control chart. Such runs indicate shift in the process level.



3. The sample points on $\bar{x}$ and R charts, too close to the central line, exhibit another form of assignable cause.



4. **Presence of Trends**. The trends exhibited by sample points on the control chart are also an indication of assignable cause a phenomenon usually observed in engineering industry, indicates the gradual shift in the process level and it is essential to determine when machine resetting becomes desirable bearing in mind that too frequent adjustments are a serious setback to production output.

6.6 SUMMARY AND FURTHER SUGGESTED READING

The success of the control chart technique depends largely upon the efficient grouping of items into samples, such that variation in quality among items within the same sample is small, but variation between one sample and another is as large as possible. Such a sample is known as 'rational sub-group'

The act of getting a process in statistical control helps in the detection and correction of many production troubles and brings about a substantial improvement in the product quality and reduction of spoilage and rework control charts is a device to achieve it. It provides better quality assurance at lower inspection cost. It is useful not only in the sphere of production, but also in other areas like packaging, scrap and spoilage, recoveries, advertising, etc.

BOOKS RECOMMENDED

1. Montgomery, D. C.: Statistical Quality Control, John Wiley and Sons, Inc., New York.

2 SP20: Handbook of SQC, Bureau of Indian Standards.

6.7 SELF ASSESSMENT QUESTIONS

1. Explain in detail the construction and working of and R charts. How control limits in these control charts are computed. What purposes do they serve.
2. Explain clearly the basis and working of control charts for mean and range.
3. In an inspection of the a machine part, the average values of 16 sub-groups were found to be

   $\overline{X}$ =0.900, $\overline{R}$ =0.028

   $A_2 = 0.58, D_3 = 0 \text{ and } D_4 = 2.11$

   Compute the control limits for and R—charts.
4. Explain bow and R charts are drawn in practice. How would you interpret the points falling outside the control limits on these charts?
5. Describe the types of control charts and criteria for this classification.

7.1 OBJECTIVES

The main objectives of this lesson are:

- To introduce the concept of control charts for variables.
- To explain the procedure of control chart for variables.
- To explain the procedure and application of control chart for the variables
- Also to provide some numerical examples of control charts for variables.

7.2 CONTROL CHARTS FOR VARIABLES

From earlier discussion as we know that amount of variation, in the produced item, is inherent in any production scheme. This Variation is the totality of numerous characteristics of the production process viz., raw material, machine setting and handling, operators, etc. This variation is the result of

(i) Chance causes and (ii) Assignable causes.

Depending on the nature of quality which is desired to control, if quality is stated in terms of a variable, e.g. diameter, tensile strength etc., we have Control Charts for Variables. Three such charts which are in common use are:

(a) Mean chart ($(\overline{X})$ -chart), (b) Range chart (R- chart),

(c) Standard Deviation chart (σ -chart)

7.3 CONTROL CHARTS FOR MEAN AND RANG

The chart is constructed on the basis of a series of samples drawn frequently during a production process, which are called ‘sub-groups’ or ‘rational sub-groups’. It is assumed that the variation between groups can be only due to chance but within groups these are due to assignable causes. Usually, smaller sub-groups of size 4 or 5 units are preferred and at least 25 such sub-groups are used in the evaluation of control limits.

The control limits in the $(\overline{X})$-chart and Range (R) charts are so placed that they reveal the presence or absence of assignable causes of variation in the

(a) **Average**—mostly related to machine setting, and

(b) **Range**—mostly related to negligence on the part of the operator.

7.4 CONSTRUCTION OF CONTROL CHARTS FOR MEAN $(\overline{X})$ & (R) RANGE

1. **Measurement of $\overline{x}$ -chart and R Charts**. Actually the work of a control chart starts first with measurements; any method of measurement has its own inherent variability. Errors in measurement can enter into the data by

(i) The use of faulty instruments,

(ii) Lack of clear-cut definitions of quality characteristics and the method of taking measurements, and

(iii) Lack of experience in the handling or use of the instrument, etc.

It is important that the mistakes in reading measurement instruments or errors in recording data should be minimized so as to draw valid conclusions from control charts.

2. **Selection of Samples or Sub-groups**: In order to make the control chart analysis effective, it is essential to pay due regard to the rational selection of the samples or sub-groups. The choice of the sample size n and the frequency of sampling i.e., the time between the selections of two groups, depend upon the process and no hard and fast rules can be laid down for this purpose. Usually n is taken to be 4 or 5 while the frequency of sampling depends on the state of the control exercised. Normally 25 samples of size 4 each or 20 samples of size 5 each under control will give good estimate of the process average and dispersion

3. **Calculation of $\overline{X}$ and R for each Sub-group**.

The mean for each sample X. Suppose we draw k independent random samples each of fixed size n from the lot to examine if the process is in a state of statistical control or not. Let us suppose that $\overline{X}_1, \overline{X}_2, ...\overline{X}_k$, be the means of the observations on the 1st, 2nd,..., kth sample respectively and let $R_1, R_2, ..., R_k$ be the values of corresponding ranges for the k samples. Symbolically,

Let Xij , j =1,2, ...,n be the measurements on the ith sample (i =l,2, ..,k). The mean X, the range R and the standard deviations for the ith sample are given by

$$\overline{X}_i = \frac{1}{n}\sum_i X_{ij} \qquad R_i = \max_j X_{ij} - \min_j X_{ij} \qquad s_i^2 = \frac{1}{n}\sum_j (X_{ij} - \overline{X}_i)^2$$

Next we find $\overline{\overline{x}}$, $\overline{R}$ and $\overline{s}$ the averages of sample means, sample ranges and sample standard deviations, respectively,

$$\overline{X} = \frac{1}{k}\sum_i \overline{X}_i , \qquad \overline{R} = \frac{1}{k}\sum_i R_i , \qquad \overline{s}_i = \frac{1}{k}\sum_i s_i$$

4. **Setting of Control Limits.** If σ is the process standard deviation then the standard error of sample mean is $\frac{\sigma}{n}$, where n is the sample size i.e., Also from the sampling distribution of range we know that

$$E[R] = d_2.\sigma$$

where d_2 is a constant depending on the sample size. Thus

$$\overline{R} = d_2.\sigma \qquad \Rightarrow \sigma = \frac{\overline{R}}{d_2} \qquad .,...........(1)$$

Also $\overline{\overline{x}}$ gives an unbiased estimate of the population mean μ, so that

$$E[\overline{\overline{x}}] = \mu \qquad ....(2)$$

5. **Control Limits for $\overline{\overline{X}}$ -chart:** when both and are known

Where $A = \frac{3}{\sqrt{n}}$

However σ , which is the standard deviation based on population is normally not available, the alternative is of using the standard deviation based on a number of small sample or one large sample .

When both μ and σ are unknown then using their estimates and a given in (1) and (2) respectively, we get the 3-σ control limits on the mean chart as

$$\bar{\bar{X}} \pm 3.\frac{\bar{R}}{d_2}\frac{1}{\sqrt{n}} = \bar{\bar{X}} \pm \left(\frac{3}{d_2\sqrt{n}}\right)\bar{R} = \bar{\bar{X}} \pm A_2\bar{R}$$

Where $A_2 = \left(\frac{3}{d_2\sqrt{n}}\right)$

Thus $UCL = \bar{\bar{X}} + A_2\bar{R}$

$LCL = \bar{\bar{X}} - A_2\bar{R}$

$CL = \bar{\bar{X}}$

Since d_2 is a constant depending on n, A_2 also depends only on n and its values have been computed and tabulated for different values of n from 2 to 25.

If control limits are to be obtained in terms of $\bar{s}$ rather than R, then an estimate of σ can be obtained from the relation

$$\sigma = \bar{s}/C_2$$

where $C_2 = \sqrt{\frac{2}{n}}.\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!}$

$$UCL_{\bar{x}} = \bar{\bar{X}} + \left(\frac{3}{\sqrt{nC_2}} \right) \bar{s} = \bar{\bar{X}} + A_1 \bar{s}$$

$$LCL_{\bar{x}} = \bar{\bar{X}} - \left(\frac{3}{\sqrt{nC_2}} \right) \bar{s} = \bar{\bar{X}} - A_1 \bar{s}$$

$$CL = \bar{\bar{X}}$$

6. Control Limits for R-chart: 3-σ control limits for R-chart are given by

$$E[R] \pm 3\sigma_R$$

E(R) is estimated by $\bar{R}$ and σ_R is estimated from the relation

$$\sigma_R = CE(R) = C\bar{R}$$

where c is a constant depending upon n and control limits are

$$UCL_R = \bar{R} + C\bar{R} = (1+C)\bar{R} = D_4\bar{R} \quad \text{and}$$

$$LCL_R = \bar{R} - C\bar{R} = (1-C)\bar{R} = D_3\bar{R}$$

$$CL_R = \bar{R}$$

The values of D_4 *and* D_3 have been tabulated for different values of n from 2 to 25. The control limits for R-chart can be obtained directly from the σ assumed or known value of a as follows:

$$UCL_R = D_2\sigma \quad \text{And}$$

$$LCL_R = D_1\sigma$$

6. **Construction of Control Chart for $\bar{X}$ and R**: Control charts are plotted on a rectangular co-ordinate axis-vertical scale (ordinate) representing the statistical measures $\bar{X}$ and R, and horizontal scale (abscissa) representing the sample number. Hours, dates or lot numbers may also be represented on the horizontal scale. Sample points mean or range is indicated on the chart by points, which may or may not be joined.

Usually the central line is drawn as a solid horizontal line and UCL and LCL are drawn at the computed values as dotted horizontal lines.

3.5 CRITERION FOR DETECTING LACK OF CONTROL CHARTS FOR MEAN $(\overline{X})$ AND (R) RANGE

For a process to be working under statistical control, points both in the and R charts should lie between the control limits. A process which is not in statistical quality control suggests the presence of assignable causes of variation which throw the process out of control. These causes must be traced and eliminated so that the process may return to operation under stable statistical conditions.

The choice between the $\overline{X}$ and the R chart is a managerial problem. ? $\overline{X}$ Chart is used to show the quality averages of the samples drawn from a given process, whereas R chart is used to show quality dispersions (variabilities) of the samples. If the presence of an assignable cause is noticed on both types of charts only one type of $\overline{X}$ chart should be used.

In practice, R charts are constructed first. If R chart indicates that the dispersion of' the quality by the process is out of control, it is better not to construct chart until the quality dispersion is brought under control.

Interpretation of $\overline{X}$ and R charts: In order to judge if a process is in control, $\overline{X}$ and R charts should be examined together and the process should be deemed in statistical control if both the charts show state of control

Interpretation of $\overline{x}$ and R charts: In order to judge if a process is in control, and R charts should be examined together and the process should be deemed in statistical control if both the charts show state of control

If a point falls beyond the control limits, it is regarded that an assignable cause has thrown the process out of control. The next step is to remove all sample results which are outside the control limits or revise the control limits for the remaining samples. Compare all the remaining plotted points against the revised control limits. If necessary, the procedure of computation may be repeated again and again until the entire sample means are within the new control limits. The new limits are then extended for checking the quality of future products.

In addition to this standard procedure following points may be helpful in detecting the lack of control:

1. If all the sample points $\bar{x}$ or *R*, as the case may be, are scattered within the control limits, the process is under control and is satisfactory. Even when all the points are within control limits concentration near the upper or the lower control limit or there is a clear upward or downward bias of points it should be checked whether the machine requires adjustment. It is also likely that the measuring gauge or the central line needs adjustment.

2. Assuming the population to be normal, the probability of any point falling outside the control limits $\pm\sigma$ or ±3R is only 0.0027 i.e., 27 in 10,000 samples, yet, if a point falls just outside the limits it will be desirable to take another bigger sample. If the mean of the combined samples still lies out of the control limits there should be search for assignable causes.

3. If one or more of the points in any or both the charts go out of the control limits we say that the process is out of control. i.e., it is not in state of statistical control. Such a situation indicates the presence of some assignable causes which must be traced, identified and eliminated. Some of the common assignable causes of variation are: defective or substandard raw material, substandard or faulty tools or equipment, inefficient or unskilled operators, negligence on the part of operators etc.

4. *A run of seven or more points*. Although all the sample points are within control limits usually the pattern of points in the chart indicates assignable causes. One such situation is a run of 7 or more points above or below the central line in the control chart. Such runs indicate shift in the process level.

5. The sample points on $\bar{x}$ and R charts, too close to the central line, exhibit another form of assignable cause.

6. *Presence of Trends*. The trends exhibited by sample points on the control chart are also an indication of assignable cause a phenomenon usually observed in engineering industry, indicates the gradual shift in the process level and it is essential to determine when machine resetting becomes desirable bearing in mind that too frequent adjustments are a serious setback to production output.

Illustration: Assuming that we know both the mean and the standard deviation of the process, that is μ and σ.

Let us consider an example; Mr. Reddy from ABC India Limited who is a Quality Control Engineer at firm knows that the mean diameter and the standard deviation of the same should be 10.5 cms and 0.02 cms respectively. Suppose he has to monitor the diameter of the product on ten consecutive days, he randomly selects seven items on each day and computes their mean (this he does as the diameter of all the pistons manufactured cannot be measured in an exhaustive manner and hence substitution of the sample mean and the standard deviation for population mean and standard deviation). Also from this data he can plot a sampling distribution, the mean and the standard deviation of the sampling distribution will be

$$\mu_{\overline{X}} = \mu = 10.5$$

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{0.02}{\sqrt{7}} = \frac{0.02}{2.645} = 0.0075$$

On plotting a chart by taking time variable (in days) on X-axis and the sample means on Y-axis, we get a chart which is called as $\overline{X}$ chart. In this chart the center line refers to the mean of the sample distribution. In addition to this, we also plot two lines called as upper and lower control limits. The lower control limit is given by

$\mu_{\overline{X}} - \sigma_{\overline{X}}$ and the upper control limit by $\mu_{\overline{X}} + \sigma_{\overline{X}}$.

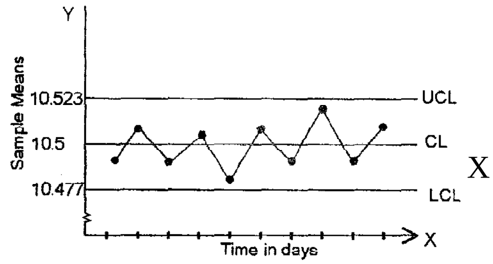
To understand why 3σ is used, let us recollect that according to Chebyshev's theorem, irrespective of the shape of the distribution, about 89 percent of the values fall within three standard deviations on either side of the sample mean and the same according to the normal distribution is 99.7 percent of the values. , the upper and lower limits are given by

$$UCL_{\overline{X}} = \mu_{\overline{X}} + 3\sigma_{\overline{X}} = \mu + 3\frac{\sigma}{\sqrt{n}} = 10.5 + 3(0.0075) = 10.523$$

$$LCL_{\overline{X}} = \mu_{\overline{X}} - 3\sigma_{\overline{X}} = \mu - 3\frac{\sigma}{\sqrt{n}} = 10.5 - 3(0.0075) = 10.477$$

$$CL_{\overline{X}} = \mu_{\overline{X}} = \mu = 10.5$$

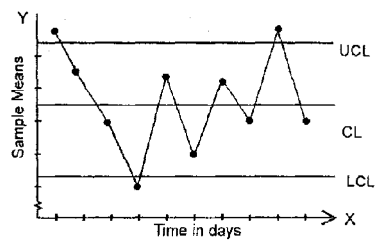
By using the sample means obtained during ten days readings, the mean of the sampling distribution and the two limits, we plot the chart and it is shown below. The values plotted are within the lower and the upper control limits indicating that the quality of output is within the acceptable range.



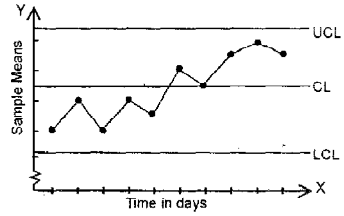
The distribution of values does not indicate any trend which may indicate a possibility of crossing the limits in the near future. Hence, when we get a plot like this we conclude that the process is in-control.

SOME IMPORTANT CRITERION FOR DETECTING LACK OF CONTROL

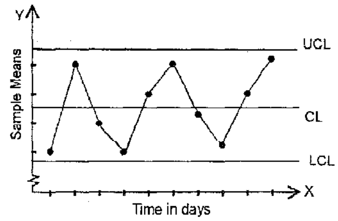
Now we look at some of the patterns we generally come across in control charts and their respective interpretations.



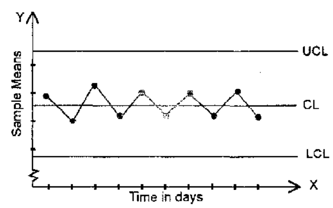
In this chart we observe that some of the points lie outside the upper and lower limits. This pattern indicates the presence of systematic variation and it should be set right first to bring the process into ‘in control’, before one plan to redesign the whole process



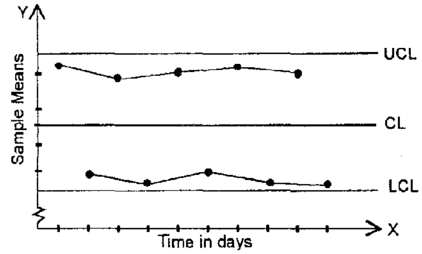
This pattern shown above is usually referred to as increasing trend. Similarly we also have decreasing trend. We observe that there is no randomness in their occurrence. This may be possibly due to the presence of some systematic variation. Although the points fall within the limits, we conclude that process is out of control



This pattern is referred to as Cycles. We observe the formation of the waves and a repetition of the pattern above and below the center line at regular intervals. This may indicate the presence of the random variation in the process



Here the deviations are minor and uniform in nature. This indicates that the variation has been reduced to a great extent. The personnel should strive to maintain this level and if deemed necessary the width of the limits should be narrowed so that further improvisation takes place in the quality of the manufacturing process.



This pattern (shown above) refers to the deviations being uniform in nature but have large magnitude. This indicates that the means of two different populations are being observed.

When the Mean and the Standard Deviation are not known

We consider the data corresponding to the above mentioned example of ABC India Limited. Since we do not know population mean and the standard deviation, we employ the sample mean as a substitute. But for this data we have ten sample means and we do not know which one truly reflects the process mean. The calculation of mean from the sample means (this is referred to as grand mean and is denoted by $\overline{\overline{X}}$), the calculation of the range and its mean are shown in the table below.

The grand mean, that is $\overline{\overline{X}}$, is calculated by employing either

$$\overline{\overline{X}} = \frac{\sum X}{n \times k} \text{ or } \overline{\overline{X}} = \frac{\sum \overline{X}}{k}$$

Here ‘n’ denotes the number of samples that are randomly selected on a day and k denotes the number of days we take the samples.

Day	Diameter (in cms)							Sample means	Sample Range
1	10.34	10.45	10.25	10.37	10.56	10.63	10.41	10.43	0.38
2	10.41	10.26	10.58	10.46	10.62	10.34	10.57	10.46	0.36
3	10.23	10.54	10.34	10.48	10.62	10.26	10.43	10.41	0.39
4	10.56	10.67	10.22	10.35	10.44	10.55	10.27	10.44	0.45
5	10,37	10.21	10.41	10.56	10.68	10.44	10.52	10.46	0.47
6	10.25	10.34	10.29	10.64	10.35	10.52	10.48	10.41	0.39
7	10.67	10.49	10.39	10.28	10.59	10.62	10.34	10.48	0.39
8	10.43	10.37	10.61	10.23	10.38	10.57	10.29	10.41	0.38
9	10.31	10.55	10.47	10.51	10.23	10.62	10.33	10.43	0.39
10	10.57	10.43	10.54	10.45	10.27	10.35	10.46	10.44	0.30
Sum	104.14	104.31	104.10	104.33	104.74	104.90	104.10	104.37	3.90
	Sum of all diameters =730.62 Mean of R=0.39								

The value of the grand mean in this problem by the first formula is given by

$$\bar{\bar{X}} = \frac{\sum X}{n \times k} = \frac{730.62}{7 \times 10} = 10.437$$ and by the second formula it is

$$\bar{\bar{X}} = \frac{\sum \bar{X}}{k} = \frac{104.437}{10} = 10.437$$

This value of $\bar{\bar{X}}$ is utilized in drawing the center line in the chart.

How to find he estimate of standard deviation

As we use sample mean as a substitute for population mean, sample range are used for standard deviation, it became a norm to employ the mean of the sample range in the calculation of the standard deviation. However, the relationship between the process standard deviation and the mean of sample range is given by a factor denoted by "d_2". This factor depends on the size of the sample "n". the upper and the lower limits we employ the following formula.

$$\overline{\overline{X}} \pm 3.\frac{\overline{R}}{d_2}\frac{1}{\sqrt{n}} = \overline{\overline{X}} \pm \left(\frac{3}{d_2\sqrt{n}}\right)\overline{R}$$ Therefore

$$UCL = \overline{\overline{X}} + \left(\frac{3}{d_2\sqrt{n}}\right)\overline{R}$$

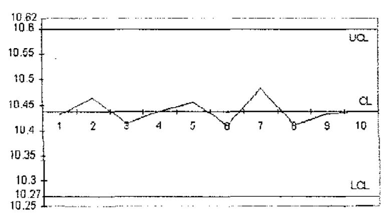
$$LCL = \overline{\overline{X}} - \left(\frac{3}{d_2\sqrt{n}}\right)\overline{R}$$

$$CL = \overline{\overline{X}}$$

Further these limits can be expressed as $UCL = \overline{\overline{X}} + A_2\overline{R}$ and $LCL = \overline{\overline{X}} - A_2\overline{R}$ denoting

Where $A_2 = \left(\frac{3}{d_2\sqrt{n}}\right)$

It is possible to look for these values directly from the tables. From the tables, the value of is equal to 2.704 for a sample size of 7. In our example the values of the lower and the upper control limits are given by



Using these values we plot the center and the upper and lower limit lines. Plot $\overline{X}$ values to get the mean chart and determine whether the process is 'in-control' or 'out-of-control'. In our problem the chart denotes that the process is in-control .Since no sample point falls outside the upper and lower control limits.

Example: A machine is set to deliver packets of a given weight. 10 samples of size 5 each were recorded. Below are given relevant data:

Sample No.	1	2	3	4	5	6	7	8	9	10
Mean ($\overline{x}$)	15	17	15	18	17	14	18	15	17	16
Range (R)	7	7	4	9	8	7	12	4	11	5

Calculate the values for the central line and the control limits for mean chart and the range chart and then comment on the state of control.

(Given: Conversion factors for n=5, $A_2 = 0.58$, $D_3 = 0$, $D_4 = 2.115$)

Solution. From the above data, we get

$$\overline{\overline{X}} = \frac{1}{10}\sum \overline{X} = \frac{15+17+15+18+17+14+18+15+17+16}{10} = \frac{162}{10} = 16.2$$

$$\overline{R} = \frac{1}{10}\sum R = \frac{7+7+4+9+8+7+12+4+11+5}{10} = \frac{74}{10} = 7.4$$

We are given for n=5,,,

Mean Chart. 3-σ Control limits for X-chart are:

$$UCL_{\overline{X}} = \overline{\overline{X}} + A_2\overline{R}.$$

$$= 16.2 + 0.58 \times 7.4 = 20.492$$

$$LCL_{\overline{X}} = \overline{\overline{X}} - A_2\overline{R}$$
$$= 6.2 - 0.58 \times 7.4 = 11.908$$

$$CL_{\overline{X}} = \overline{\overline{X}} = 6.2$$

Since the entire sample points (means) fall within the control limits, this chart shows that the process is in the state of statistical control.

Range Chart (R-chart): 3-σ control limits for R-chart are

$$UCL_R = D_4\overline{R} = 2.115 \times 74 = 15.614$$

$$LCL_R = D_3\overline{R} = 0 \times 74 = 0$$

$$CL_R = \overline{R} = 7.4$$

As the entire sample points (ranges) lie within the control limits, R-chart also shows that process is in control.

Since both $\overline{x}$ and R charts depict control, the process can be regarded to be in statistical control, in other words the process is operating free from assignable causes of variations and under the influence of only chance causes of variation.

7.6 SUMMARY AND FURTHER SUGGESTED READING

The main objectives of this was to introduce the concept of control charts for variables and to explain the procedure of control chart for variables, its applications by providing some numerical examples of control charts for variables.

The main object of the control chart is to indicate when a process is not in control. The criteria for detecting lack of control are, therefore, of fundamental and crucial importance. The control limits in the $(\overline{X})$-chart and Range (R) charts are so placed that they reveal the presence or absence of assignable causes of variation in the a**verage(m**ostly related to machine setting) and **Range**(mostly related to negligence on the part of the operator.)

BOOKS RECOMMENDED

1. Montgomery, D. C. : Statistical Quality Control, John Wiley and Sons, Inc., New York.

2 SP20 : Handbook of SQC, Bureau of Indian Standards.

3. Fundamental of applied statistics by S.C Gupta and V.K Kapoor ,S.Chand New Delhi

7.7 SELF ASSESSMENT QUESTIONS

Question 1: What is control chart? Explain the basic principles underlying the control charts. Discuss the role of control charts in manufacturing processes.

Question 2: Explain the justification for using the three sigma limits in the control charts irrespective of the actual probability distribution of the quality characteristic.

Question 3: Explain clearly the basis and working of control charts for mean and range. State the basis and assumptions on which and R charts are developed.

Question 4: Construct a control chart for mean and the range for the following data on the basis of fuses, samples of size 5 being taken every hour (each set of 5 has been arranged in ascending order of magnitude).

42	42	19	36	42	51	60	18	15	69	64	61
65	45	24	54	51	74	60	20	30	109	90	78
75	68	80	69	57	75	72	27	39	113	93	94
78	72	81	77	59	78	95	42	62	118	109	109
87	90	81	84	78	132	138	60	84	153	112	136

Comment on whether the production is under control or not.

8.1 OBJECTIVES

The main objectives of this lesson are:

- To introduce the concept of control charts for attributes.
- To explain the procedure of control chart for fraction defectives.
- To provide the students the knowledge about the need of control chart for number of defectives
- To explain the procedure and application of control chart for the number of defects, and
- Also to provide some numerical examples of control charts for attributes to illustrate its practical use.

8.2 CONTROL CHARTS FOR ATTRIBUTES OBJECTIVES AND INTRODUCTION

Sometimes the characteristic representing the quality of a product is discrete, such as number of surface defects in floor, number of defective items in a lot, etc. In such cases, the number of defects in an item may be nil, one, two or more. In such cases, the distribution explaining the number of items according to number of defects on it, when the process is in control will be a Poisson distribution. Under such a circumstance control charts for average number of defects per item (C-chart) is used.

In some such cases the use of is $(\overline{X})$ and R is restricted due to the following limitations

1. They are charts for variables only i.e., for quality characteristics which can be measured and expressed in numbers.

2. 2. In certain situations they are impracticable and un-economical.

As an alternative to $(\overline{X})$ and R charts, we have the control chart for attributes which can be used for quality characteristics.

For example, very often articles are inspected or tested in batches and classified either as 'good' (i.e. those which conform to specifications) or 'defective' (i.e. those which are considered as useless rejections). The quality of a product is then measured by "fraction defective" $p = \frac{d}{n}$, where p is a fraction and d is the number of defectives found in a sample of n articles inspected. Again, a single product or a specified size of the product may contain one or more defects, e.g. paper, textiles, metallic sheets, pipes etc. The number of defects c is then used as a quality measure. Two important control charts for) attributes are

(a) Fraction Defective Chart (p chart)

(b) Number of Defectives chart (np chart)

(c) Number of Defects chart (c chart)

8.2 CONTROL CHARTS FOR FRACTION DEFECTIVES (p-chart)

The control chart for defectives, known as p-Chart is used whenever the quality characteristic observed in the classification of items as defective or not-defective is the result of inspection of castings, go and not-go gauge test results, etc. The objective of this chart is to evaluate the quality of the items (that is the average fraction defective or per cent defective) and to note the changes in quality over a period of time. The chart can also be used for routine control and is an immediate guide for correcting causes of bad quality. Also in many situations these charts suggest where X^{-} and R charts could be used.

The concept of rational sub-groups plays an important part in the interpretation of p-Chart also. Thus, the inspection results by different inspectors, fraction defectives of different machines doing the same job, or of different shifts, have to be charted separately until the evidence shows that the performance of machines, inspectors and different shifts are the same.

Generally preliminary data for the construction of p-chart are obtained from the past records. About 20 to 25 samples may be sufficient to get an idea of the working of the process and to evaluate the standard fraction defective for future control.

To know whether a process is in state of control or not in necessary to calculate the trial control limits.

While dealing with attributes, a process will be adjudged in statistical control if all samples or sub-groups are ascertained, i have the same population proportion P.

If 'd' is the number of defectives in a sample of size n then sample proportion defective is p=$\frac{d}{n}$. Hence d is a binomial variate with parameters n and P.

$$\therefore \quad E(d) = nP \quad and \quad Var(d) = nPQ, \qquad Q = 1 - P$$

$$\therefore \quad E(p) = E\left(\frac{d}{n}\right) = \frac{1}{n}E(d) = P \quad and$$

$$Var\left(\frac{d}{n}\right) = \frac{1}{n^2}Var(d) = \frac{1}{n^2}.nPQ = \frac{PQ}{n}$$

Thus, Thus 3-σ control limits for p-chart are given by

$$E[p] \pm 3S.E(p) = P \pm 3\sqrt{PQ/n} = P \pm A\sqrt{PQ}$$

A= $3/\sqrt{n}$ has been tabulated for different values of n.

If P' is the given or known value of P

$$UCL_p = P' + A\sqrt{P'(1-P')}$$

$$LCL_p = P' - A\sqrt{P'(1-P')}$$

$$CL_p = P'$$

If P is unknown, Let d be the number of defectives and P: the fraction defective for the ith sample (i= 1,2, .., k) of size n. Then the population proportion P is estimated by the statistic $\overline{p}$ given by

$$\overline{p} = \frac{\sum d_i}{\sum n_i}$$

where is an unbiased estimate of P and control limits are given by

$$UCL_p = \overline{p} + A\sqrt{\overline{p}(1-\overline{p})}$$

$$LCL_p = \overline{p} - A\sqrt{\overline{p}(1-\overline{p})}$$

$$CL_p = \overline{p}$$

If all the sample points fall within the control limits without exhibiting any specific pattern, the process is said to be in control. In such a case, the observed variations in the fraction defective are attributed to the stable pattern of chance causes.

Illustration: A team in an accounting group is trying to reduce the cost of processing invoices by decreasing the fraction of invoices with errors. The team decided to take a random sample of 100 invoices per day. If the invoice had one or more errors it was defective. The data from the last 25 days is taken. The data obtained after 25 days is as under

Day	Invoice inspected	Number of defective(np)	Fraction Defective(p)
1	100	22	0.22
2	100	33	0.33
3	100	24	0.24
4	100	20	0.2
5	100	18	0.18
6	100	24	0.24
7	100	24	0.24
8	100	29	0.29
9	100	18	0.18
10	100	27	0.27
11	100	31	0.31
12	100	26	0.26
13	100	31	0.31
14	100	24	0.24
15	100	22	0.22
16	100	22	0.22
17	100	29	0.29
18	100	31	0.31
19	100	21	0.21
20	100	26	0.26
21	100	24	0.24
22	100	32	0.32
23	100	17	0.17
24	100	25	0.25
25	100	21	0.21

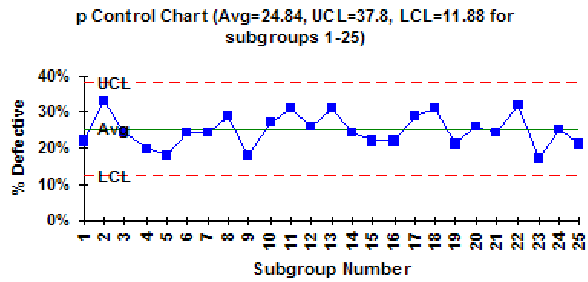
The subgroup size is n = 100 $\bar{p} = 0.248 = 24.8\%$, $\bar{n} = \frac{\sum n}{k} = \frac{2500}{25} = 100$

Since the subgroup size is constant, the average subgroup size is 100. The next step is to calculate the control limits. The control limits calculations are shown below.

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}\bar{q}}{\bar{n}}} = 0.248 + 3\sqrt{\frac{(0.248)(1-0.248)}{100}} = 0.378 = 37.8\%$$

$$LCL_p = \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{\overline{n}}} = 0.248 - 3\sqrt{\frac{(0.248)(1-0.248)}{100}} = 0.118 = 11.8\%$$

$$CL_p = \overline{p} = 0.248$$



The values of p, the average, and the control limits are plotted in the figure. Try to find the answer of the following questions.

1. What variation is the p control chart examining?

2. Is the process in statistical control? What does this mean?

3. What other type of statistical tool could be used in conjunction with this p control chart and why?

8.2 CONTROL CHARTS FOR NUMBER OF DEFECTIVES (d-chart).

A d chart shows the actual number of defectives found in each sample. If the number of items inspected on each occasion is the same, the plotting of the actual number of defective may be more convenient and meaningful than the fraction defective. The construction and interpretation of the number defective chart called np-chart is similar to that of the p-chart. The difference is that the actual number of defective (np) in samples of fixed size (n) is plotted instead of the fraction defective (p) and the central line is drawn at np instead of p.

Here instead of p, the sample proportion defective, we use d, the number of defectives in the sample, then the 3-σ control limits for d-chart are given by

$$E[d] \pm 3S.E(d) = nP \pm 3\sqrt{nP(1-P)}$$

If P' is the given or known value of P

$$UCL_d = nP' + 3\sqrt{nP'(1-P')}$$

$$LCL_d = nP' - 3\sqrt{nP'(1-P')}$$

$$CL_d = nP'$$

If P is unknown in this case an unbiased estimate the population proportion P is given by the statistic $\bar{p}$ consequently 3- control limits are

$$UCL_d = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$$

$$LCL_d = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$$

$$CL_d = n\bar{p}$$

Fixed Sample Size. If the sample size remains constant for each sample

i.e.,if $n_1 = n_2 .. = n_k = n$, (say) then an estimate of the population proportion P is given by

$$\hat{P} = \bar{p} = \frac{\sum_{i=1}^{k} np_i}{\sum_{i} n} = \frac{n\sum_{i=1}^{k} p_i}{nk} = \frac{1}{k}\sum_{i=1}^{k} p_i$$

If the entire sample points fall within the control limits without exhibiting any specific pattern, the process is said to be in control. In such a case, the observed variations in the fraction defective are attributed to the stable pattern of chance causes

8.5 CONTROL CHARTS FOR NUMBER OF DEFECTS(c-chart)

The c-chart is used for the control of number of defects per unit. Although the application of the c-chart is somewhat limited, compared with the p-chart, there are many instances in industry where it is useful. For example; in the control of number of defects in material, number of soiled packages in a given consignment, number of weak spots in a given length of wire, number of errors made by a worker during a given period of time, etc.

The sample size for the c-chart, is usually, any one of the fixed time, length, area, a single unit, or group of units.

The pattern of variation for the counted number of defects can be represented by the Poisson distribution. When the probability (p) of defect is very small, and the sample size (n) very large such that np is a finite number, the probability of obtaining 0, 1, 2,... defects is given by the terms of the Poisson distribution.

For instance in many manufacturing or inspection situations, the sample size n i.e., the area of opportunity is very large (since the opportunities for defects to occur are numerous) and the probability p of the occurrence of a defect in any one spot is very small such that np is finite, In such situations the pattern of variations in data can be represented by Poisson distribution, and consequently 3-σ control limits based on Poison distribution are used. Since for a Poisson distribution mean and variance are equal, if we assume c that is Poisson variate with parameter, λ, we get

$$E[c] = \lambda \quad Var(c) = \lambda$$

Thus 3-σ control limits for c-chart are given by

$$UCL_c = E(c) + 3\sqrt{Var(c)} = \lambda + 3\sqrt{\lambda}$$

$$LCL_c = E(c) - 3\sqrt{Var(c)} = \lambda - 3\sqrt{\lambda}$$

$$CL_c = \lambda$$

Standards not given: If the value of λ is not known, it is estimated by the mean number of defeats per unit. Thus, if c is the number of defects observed on the ith (i=1,2.. k) inspected unit, then an estimate of λ is given by,

$$\hat{\lambda} = \bar{c} = \sum_{i=1}^{k} \frac{c_i}{k}$$

It can be easily seen that $\bar{c}$ is an unbiased estimate of λ control limits in this case, are given by

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}}$$

$$CL_c = \bar{c}$$

Uses of c chart

In spite of the limited field of application of c-chart (as compared to X, R, p-charts), there do exist situations in industry where c-chart is definitely needed. Some of the situations to which c-chart can be applied with advantage are:

1. C is number of imperfections observed in a bale of cloth.
2. C is the number of surface defects observed in (i) roll of coated paper or a sheet of photographic film, and (ii) a galvanised sheet or a painted, plated or enameled surface of given area.
3. C is the number of defects of all types observed in aircraft sub-assemblies or final assembly.
4. C is the number of breakdowns at weak spots in insulation in a given length of insulated wire subject to a specified test voltage.
5. C is the number of defects observed in stains or blemishes on a surface.
6. C is the number of soiled packages in a given consignment.
7. C-chart has been applied to sampling acceptance procedures based on number of defects per unit, e.g., in case of inspection of fairly complex assembled units such as TV. sets, aircraft engines, tanks, machine-guns, etc., in which there are very many opportunities for the occurrence of defects of various types and the total number of defects of all types found by inspection is recorded for each unit.
8. c-chart technique can be used with advantage in various fields other than industrial quality control, e.g., it has been applied

(i) to accident statistics (both of industrial accidents and highway accidents),

(ii) chemical laboratories, and

(iii) in epidemiology.

ILLUSTRATIONS

1. Calculate the control limits for the following data and state your conclusion

Sample No: (each of 100 items)	1	2	3	4	5	6	7	8	9	10
No of defectives:	12	10	6	8	9	9	7	10	11	8

Sol: to draw our conclusion about the state of control for Fraction Defective Chart (p chart) can be used.

Here P is unknown. Let d be the number of defectives and P: the fraction defective for the ith sample (i= 1,2, .., k) of size n. Then the population proportion P is estimated by the statistic $\overline{p}$ given by

$$\overline{p} = \frac{\sum d_i}{\sum n_i} = \frac{\sum d_i}{nk} \qquad \text{(fixed sample size)}$$

where $\overline{p}$ is an unbiased estimate of P and control limits are given by

$$UCL_p = \overline{p} + A\sqrt{\overline{p}(1-\overline{p})}$$

$$LCL_p = \overline{p} - A\sqrt{\overline{p}(1-\overline{p})}$$

$$CL_p = \overline{p}$$

Where A= $3/\sqrt{n}$ has been tabulated for different values of n.

Here $\overline{p} = \frac{12+10+6++8+9+9+7+10+11+8}{10\times100}$

$$\frac{90}{1000} = 0.09 \qquad\qquad \Rightarrow \bar{q} = 1 - 0.09 = 0.91$$

Therefore control limits for fraction defective p are given by

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}\bar{q}}{n}} = 0.09 + 3\sqrt{\frac{(0.09)(0.91)}{100}} = 0.09 + 0.086 = 0.176$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}\bar{q}}{n}} = 0.09 - 3\sqrt{\frac{(0.09)(0.91)}{100}} = 0.09 - 0.086 = 0.004$$

$$CL = \bar{p} = 0.09$$

If all the sample points fall within the control limits without exhibiting any specific pattern, the process is said to be in control. In such a case, the observed variations in the fraction defective are attributed to the stable pattern of chance causes

(Graphical representation of fraction defective on control chart by using above calculated control limits is left as an exercise for the students)

2. using the data given below calculate the center line, upper control limit, lower control limits.

$n = 4 \quad \bar{R} = 9.60, \qquad d_2 = 2.059 \quad and \quad d_3 = 0.880$

In a certain sampling inspection , the number of defectives found in 10 samples of size 100 each are as given below

11 18 11 18 21 10 20 18 17 21

Do these indicate that the quality characteristic under inspection is under statistical control?

Sol:

Here instead of p, the sample proportion defective, we use d, the number of defectives in the sample, then the 3-σ control limits for d-chart are given by

$$E[d] \pm 3S.E(d) = nP \pm 3\sqrt{nP(1-P)}$$

Here P is unknown in this case an unbiased estimate the population proportion P is given

$$np = \frac{11+18+11+18+21+10+20+18+17+21}{10} = \frac{170}{10} = 17$$

$$p = \frac{170}{100 \times 10} = 0.17 \qquad q = 1 - 0.17 = 0.83$$

$$\sqrt{npq} = \sqrt{100 \times 0.17 \times 0.93} = 3.76$$

Therefore control limits are

$$UCL = 17 + 3 \times 3.76 = 28.28$$
$$LCL = 17 - 3 \times 3.76 = 5.72$$
$$CL = 17$$

Conclusion: since all the sample points are with in the control limits 5.72 to 28.28 so the quality characteristic under inspection is under statistical control.

3**.** Rolls of papers are inspected for defects. The numbers of defects as found in an inspection of 20 rolls are as under.

12,6,18,4,5,9,4,1,12,14,8,11,14,21,21,10,12,9,13,10

Calculate the control limits of an appropriate chart?

Here we are dealing with number of defects rather than number of defectives. C chart will be appropriate for defects per unit. Control limits are given by

$$E(c) \pm 3\sqrt{Var(c)} = \lambda \pm 3\sqrt{\lambda}$$

Here the value of λ is not known, it is estimated by the mean number of defeats per unit. Thus, if c is the number of defects observed on the ith (i=1,2.. k) inspected unit, then an estimate of λ is given by,

$$\hat{\lambda} = \bar{c} = \sum_{i=1}^{k} \frac{c_i}{k}$$

Control limits in this case, are given by

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}}$$

$$CL_c = \bar{c}$$

$$\bar{C} = \frac{12+6+18+4+5+9+4+1+12+14+8+11+14+21+21+10+12+9+13+10}{20}$$

$$\frac{214}{20} = 10.7$$

$$\sqrt{\bar{C}} = \sqrt{10.7} = 3.27$$

Control limits for C chart are

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 10.7 + 3 \times 3.27 = 20.51$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 10.7 - 3 \times 3.27 = 0.89$$

$$CL_c = 10.7$$

Example 4. The following data refer to visual defects found during inspection of the first 10 samples of size 100 each. Use them to obtain upper and lower control limits for percentage defective in samples of 100.

Sample No:	1	2	3	4	5	6	7	8	9	10
Number of Defectives:	4	8	11	3	11	7	7	16	12	6

Solution. (i) Total number of' defectives in 10 samples of 100 each is

$$\sum d == 4+8+ll+3+11+7+7+16+12+6 = 85$$

(i) Average fraction defective

$$\bar{p} = \frac{Total\ no.\ of\ defective}{Total\ no.\ of\ items\ inspected} = \frac{85}{10 \times 100} = 0.085$$

(iii) The 3-σ control limits for np-chart are

$$n\bar{p} \pm 3\sqrt{n\bar{p}\bar{q}}$$

$$= 100 \times 0.085 \pm 3\sqrt{100 \times 0.085 \times (1 - 0.085)}$$
$$= 8.5 \pm 2.79 = 8.5 \pm 8.37$$

Hence

$$UCL_{np} = 8.5 + 8.37 = 16.87$$

$$LCL_{np} = 8.5 - 8.37 = 0.13$$

$$CL_{np} = 8.5$$

The control chart for the number of defective units is obtained on plotting the number of defectives against the corresponding sample number and is given below

8.6 SUMMARY AND FURTHER SUGGESTED READING

In some cases $(\overline{X})$ and R are impracticable and un-economical as an alternative to and R charts, we have the control chart for attributes which can be used for quality characteristics ,very often articles are inspected or tested in batches and classified either as 'good' or 'defective'. The quality of a product is then measured by "fraction defective". Again, a single product or a specified size of the product may contain one or more defects, e.g. paper, textiles, metallic sheets, pipes etc. The number of defects c is then used as a quality measure. Three important control charts for) attributes are

(a) Fraction Defective Chart (p chart)

(b) Number of Defectives chart (np chart)

(c) Number of Defects chart (c chart

BOOKS RECOMMENDED

1. Montgomery, D. C. : Statistical Quality Control, John Wiley and Sons, Inc., New York.

2. SP20 : Handbook of SQC, Bureau of Indian Standards.

3. Fundamental of applied statistics by S.C Gupta and V.K Kapoor ,S.Chand New Delhi

8.7 SELF ASSESSMENT QUESTIONS

Q 1 Explain clearly the basis and working of control charts for fraction defectives and and number of defectives. State the basis and assumptions on which these charts are developed.

Q 2 Differentiate clearly between defective and defect.

Q3 Each day a sample of 50 items from a production process was examined. The number of defectives found in each sample was as follows:
6, 2, 5, 1, 2, 2, 3, 5, 3, 4, 12, 4, 4, 1, 3, 5, 4, 1, 4, 3, 5, 4, 2, 3.

Draw a suitable control chart and check for control.

Q4 A process produces 3% defectives. If n=4, what would be 3-σ limits for the p-chart ? Discuss the theoretical basis of p and np-charts.

UNIT- III

Lesson- 9

Lesson - 10

Lesson - 11

9.1 Objectives

The main objectives of this lesson are

- To introduce the students with the process capability studies.
- To introduce them with the various techniques used for process capability studies.
- To make the students more familiar with the concepts of process and product control and theory of control charts in its application part.

9.2 PROCESS CAPABILITY STUDIES AN INTRODUCTION

PROCESS CAPABILITY

A process is a unique combination of tools, materials, methods, and people engaged in producing a measurable output; for example a manufacturing line for machine parts. All processes have inherent statistical variability which can be evaluated by statistical methods.

The output of a process is expected to meet customer requirements, specifications, or engineering tolerances. Engineers can conduct a process capability study to determine the extent to which the process can meet these expectations.

The ability of a process to meet specifications can be expressed as a single number using a process capability index or it can be assessed using control charts. Either case requires running the process to obtain enough measurable output so that engineering is confident that the process is stable and so that the process mean and variability can be reliably estimated.

The Process Capability is a measurable property of a process to the specification, expressed as a process capability index (e.g., Cpk or Cpm) or as a process performance index (e.g., Ppk or Ppm). The output of this measurement is usually illustrated by a histogram and calculations that predict how many parts will be produced out of specification (OOS).

Process capability is also defined as the capability of a process to meet its purpose as managed by an organization's management and process definition structures

Two parts of process capability are:

(1) Measure the variability of the output of a process, and

(2) Compare that variability with a proposed specification or product tolerance.

MEASURE THE PROCESS

The input of a process usually has at least one or more measurable characteristics that are used to specify outputs. These can be analyzed statistically; where the output data shows a normal distribution the process can be described by the process mean (average) and the standard deviation.

A process needs to be established with appropriate process controls in place. A control chart analysis is used to determine whether the process is "in statistical control". If the process is not in statistical control then capability has no meaning. Therefore the process capability involves only common cause variation and not special cause variation.

A batch of data needs to be obtained from the measured output of the process. The more data that is included the more precise the result, however an estimate can be achieved with as few as 17 data points. This should include the normal variety of production conditions, materials, and people in the process. With a manufactured product, it is common to include at least three different production runs, including start-ups.

It is customary to take the six sigma $(6-\sigma)$ spread in the distribution of the quality characteristic as a measure of the process capability.

The process mean (average) and standard deviation are calculated. With a normal distribution, the "tails" can extend well beyond plus and minus three standard deviations, but this interval should contain about 99.73% of production output. Therefore for a normal distribution of data the process capability is often described as the relationship between six standard deviations and the required specification.

9.3 USE OF PROCESS CAPABILITY STUDIES

Statistical process control defines techniques to properly differentiate between stable processes, processes that are drifting (experiencing a long-term change in the mean of the output), and processes that are growing more variable. Process capability indices are only meaningful for processes that are stable (in a state of statistical control).

Process capability analysis is a vital part of an overall quality improvement program. Among the major uses of data from a process capability analysis are the following:

(i) Predicting how well the process will hold the tolerances.

(ii) Assisting product developers in selecting or modifying a process.

(iii) Assisting in establishing an interval between sampling for process monitoring.

(iv) Specifying performance requirements for new equipments.

(v) Selecting between competing vendors.

(vi) Planning the sequence of production processes when there in an interactive effect of processes on tolerances.

(vii) Reducing the variability in a manufacturing process.

Therefore, process capability analysis is a technique that has application in many segments of the product cycle, including product and process design, vendor sourcing, production planning and manufacturing.

9.4 TECHNIQUES USED IN PROCESS CAPABILITY ANALYSIS

Techniques used in Process Capability Analysis

Generally three primary techniques are used in a process capability analysis; namely, histograms or probability plots, control charts and designed experiments.

9.5 HISTOGRAM TECHNIQUE

Histogram Technique: The histogram can be used in estimating process capability. Alternatively, a stern-and-leaf diagram may be substituted for the histogram. At least 100 more observations should be available for the histogram (or the stem- and-leaf diagram) to be moderately stable so that a reasonable reliable estimate of process capability may be obtained. If the quality engineer has access to the process and can control the data collection effort, the following steps should be followed prior to data collection:

Choose the machine or machines to be used. If the results based on one machine or few machines are to be extended to a larger population of machines, then the selected machines should be representative of those in the population.

(ii) Select the process operating conditions carefully define conditions, cutting speeds, feed rates, and temperatures, for future reference. It may be important to study the effects of varying these factors on process capability.

(iii) Select a representative operator. In some studies, it may be important to estimate operator variability. In these cases, the operators should be selected at random from the population of operators.

(iv) Carefully monitor the data-collection process, and record the time order in which unit is produced.

The histogram, along with the sample average $\overline{X}$ and sample standard deviation and provides information about process capability. Below given is illustrate the histogram technique.

ILLUSTRATION: The data on Bursting Strengths for 100 one litre glass soft drink bottles are given in the following table. Use histogram technique to estimate process capability.

265	197	346	280	265	200	221	265	261	278
205	286	317	242	254	235	176	262	248	250
263	274	242	260	281	246	248	271	260	265
307	243	258	321	294	328	263	245	274	270
220	231	276	228	223	296	231	301	337	29S
268	267	300	250	260	276	334	280	250	257
260	281	208	299	308	264	280	274	278	210
234	265	187	258	235	269	265	253	254	280

299	214	264	267	283	235	272	287	274	269
215	318	271	293	277	290	283	258	275	251

Here, First of all we present the given data in the form of frequency distribution and then determine mean and standard deviation.

Class Intervals	Frequency f	Mid value (X)	$d = \frac{x - A}{h}$	fd	d^2	fd^2	Relative Frequency.	Cumulative Relative Frequency
170-190	2	180	-4	-8	16	32	0.02	0.02
190-210	4	200	-3	-12	9	36	0.04	0.06
210-230	7	220	-2	-14	4	28	0.07	0.13
230-250	13	240	-1	-13	1	13	0.13	0.26
250-270	32	260	0	0	0	0	0.32	0.58
270-290	24	280	1	24	1	24	0.24	0.82
290-310	11	300	2	22	4	44	0.11	0.93
310-330	4	320	3	12	9	36	0.04	0.97
330-350	3	340	4	12	16	48	0.03	1.00
Total	100			23		261	1.00	

Here $\bar{x} = 260 + 20 \times \frac{\sum fd}{\sum f}$

$$\bar{x} = 260 + 20 \times \frac{23}{100} = 264.60$$

$$Standard\ deviation = 20\sqrt{\frac{1}{99}[261 - 100(0.23)]}$$

$$= 20\sqrt{\frac{255.71}{99}} = 20 \times 1.61$$

$$= 32.20$$

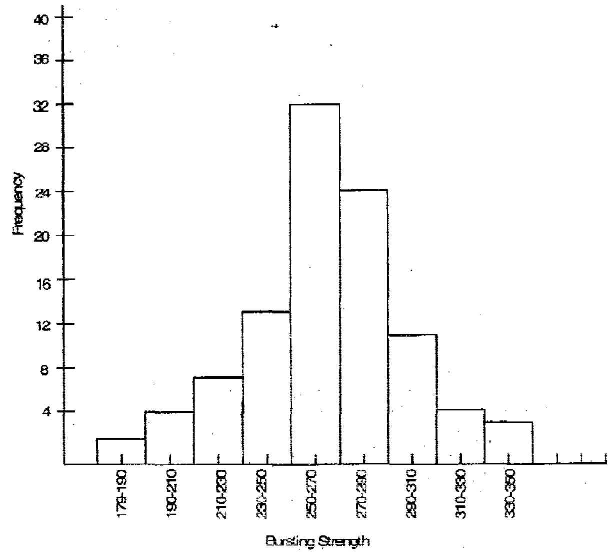
Therefore, the process capability would be estimated as

$$\bar{x} \pm 3s$$

Or $264.60 \pm 3 \times 32.20$

Or 264.60 ± 96.60

Now we plot the histogram as



From the shape of the histogram we may imply that the distribution of bursting strength is approximately normal. Thus, we can estimate that approximately 99.73% of bottles produced by this process will burst between 264.60 ± 96.60 or 168 and 361.

9.6 CONTROL CHART TECHNIQUE

Histograms summarize the performance of the process. They do not necessarily display the potential capability of the process because they do not address the issue of statistical control, or show systematic patterns in process output. Control charts are very effective in this regard. The control chart should be regarded as a primary technique of process capability analysis. Both variables and attributes control charts can be used in process capability analysis. The $\overline{X}$ and R charts should be used whenever possible, because of the greater power and better information they provide relative to attributes chart& Also, $\overline{X}$ and R charts allow us to study the processes without regard to specifications. Further $\overline{X}$ and R control charts allow both the instantaneous variability and variability across time to be analysed. It is particularly helpful if the data far a process capability study are collected in two to three different time periods. Now we summaries the use of $\overline{X}$ and R chart in the form of an illustration as give below.

Illustration: The following table presents the soft drink bottle bursting strength data in 20 samples of five observations each.

Sample	Observations					$\overline{X}_i$	Range R
1	265	205	263	307	220	252.00	102
2	268	260	234	299	215	255.20	84
3	197	286	274	243	231	246.20	89
4	267	281	265	214	318	269.00	104
5	346	317	242	258	276	287.80	104
6	300	208	187	264	271	246.00	113
7	280	242	260	321	228	266.20	93
8	250	299	258	267	293	273.40-	49
9	265	254	281	294	223	263.40	71
10	260	308	235	283	277	272.60	73
11	200	235	246	328	296	261.00	128
12	276	264	269	235	290	266.80	55
13	221	176	248	263	231	227,80	87
14	334	280	265	292	283	286.80	69
15	265	262	271	245	301	268.80	56
16	280	274	253	287	258.	270.40	34
17	261	248	260	274	337	276.00	89
18	250	278	254	274	275	266.20	48
19	278	250	265	270	298	272.20	48
20	257	210	280	269	257	253.40	70

The calculations for $\overline{X}$ and R charts are summarized as given below

$$\overline{\overline{X}} = \frac{1}{n}\sum_{i=1}^{n}\overline{X}_i = \frac{5281.20}{20} = 264.06$$

$$\overline{R} = \frac{1}{n}\sum_{i=1}^{n}R_i = \frac{1546}{20} = 77.30$$

The control limits for $\overline{X}$ -charts are

$UCL = \overline{\overline{X}} + A_2\overline{R}$ and $LCL = \overline{\overline{X}} - A_2\overline{R}$

Where $A_2 = \left(\frac{3}{d_2\sqrt{n}}\right)$

$$\begin{aligned} UCL &= 264.06 + (0.577)(77.30) \\ &= 264.06 + 44.60 \\ &= 308.66 \end{aligned}$$

$$CL = \overline{\overline{X}} = 264.06$$

$$\begin{aligned} UCL &= 264.06 - (0.577)(77.30) \\ &= 264.06 - 44.60 \\ &= 219.46 \end{aligned}$$

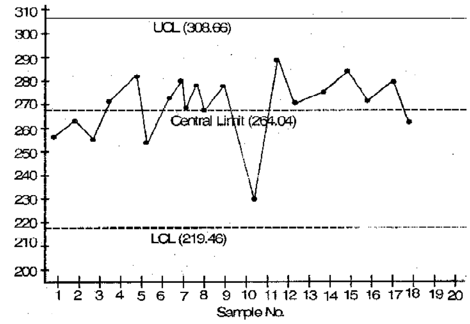
Also, the control limits for R-Chart are

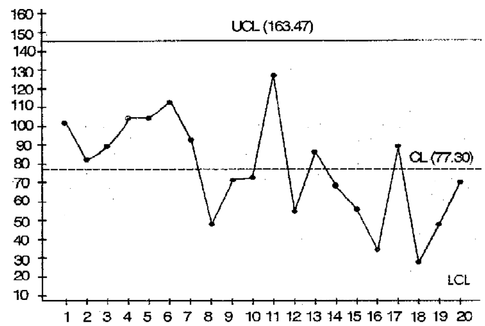
$$UCL = D_4\,(2.115)(77.30) = 163.49$$

$$C.L. = \overline{R} = 77.30$$

$$L.C.L = D_3\overline{R} = (0)(77.30) = 0$$

The values of A_2, D_3 *and* D_4 are given in the tables for given value of n. Next we plot the data on charts on the basis of control limits calculated above.





The process parameters may be estimated from the control chart as

$$\hat{\mu} = \overline{\overline{X}} = 264.06$$

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{77.30}{2.32} = 33.23$$

Thus, the one-sided lower process capability ratio is estimated by

$$\hat{C}_{pl} = \frac{\hat{\mu} - LSL}{3\hat{\sigma}}$$

$$= \frac{264.06 - 200}{3 \times 33.23}$$
$$= 0.64$$

Clearly, since strength is a safety-related parameter, the process capability is inadequate.

This example illustrates a process that is in control but operating at an unacceptable level. There is no evidence to indicate that the production of defective units is operator-controllable. Thus management intervention is required either to improve the process or to change the requirements if the quality problems with the bottles are to be solved. The main objective of these interventions is to increase the process capability ratio to at least a minimum acceptable level.

Sometimes the process capability analysis indicates an out-of-control process. It is unsafe to estimate process capability. In such cases, the process must be stable in order to produce a reliable estimate of process capability.

9.7 DESIGNED EXPERIMENT TECHNIQUE

The design of experiment is a systematic approach to varying the input controllable variables in the process and analyzing the effects of these process variables on the output. Designed experiments are also useful in discovering which set of process variables are influential on the output, and at what levels these variables should be held to optimize the process performance. Thus, design of experiments is useful in more general manufacturing and development problems than merely estimating process capability. One of the major uses of designed experiments is in isolating and estimating the sources of variability in a process.

9.8 SUMMARY AND FURTHER SUGGESTED READING

In this lesson we have explained the meaning and applications of process capability studies. The process capability analysis defined as an engineering study to estimate process capability. There are three techniques, usually used in process capability studies. These are; Histogram technique, Control charts technique and Designed experiment technique.

FURTHER SUGGESTED READINGS

1. D.C. Montgomery Introduction to Statistical Quality Control.

9.9 SELF ASSESSMENT QUESTIONS

1. What do you understand by process capability analysis?
2. Define process capability when the process shows a state of control. Discuss in detail

the various courses of action often employed under the following situations:

(i) The process capability is greater than the specified tolerance.

(ii) The process capability is approximately equal to the specified tolerance.

(iii) The process capability is less than the specified tolerance.

(iv) When only a single specification limit is given, discuss its relationship with the process capability and the actions that can be taken under different settings of the machine.

3 Explain the different methods of process capability analysis.

4. Perform a process capability analysis using (i) Histogram technique and (ii) Control charts technique by taking some suitable data.

10.1 OBJECTIVES

After the successful Completion of this lesson the student will able to understand

· The concept of acceptance sampling

· The need for the technique of acceptance sampling

10.2 AN INTRODUCTION TO PRINCIPLES OF ACCEPTANCE SAMPLING

ACCEPTANCE SAMPLING

Assume that a consumer receives a shipment of parts called a lot from a producer. A sample of parts will be taken and the number of defective items counted. If the number of defective items is low, the entire lot will be accepted. If the number of defective items is high, the entire lot will be rejected. Correct decisions correspond to accepting a good-quality lot and rejecting a poor-quality lot. Because sampling is being used, the probabilities of erroneous decisions need to be considered. The error of rejecting a good-quality lot creates a problem for the producer; the probability of this error is called the producer's risk. On the other hand, the error of accepting a poor-quality lot creates a problem for the purchaser or consumer; the probability of this error is called the consumer's risk.

The design of an acceptance sampling plan consists of determining a sample size n and an acceptance criterion c, where c is the maximum number of defective items that can be found in the sample and the lot still be accepted. The key to understanding both the producer's risk and the consumer's risk is to assume that a lot has some known percentage of defective items and compute the probability of accepting the lot for a given sampling plan. By varying the assumed percentage of defective items in a lot, several different sampling plans can be evaluated and a sampling plan selected such that both the producer's and consumer's risks are reasonably low.

THE NEED FOR SAMPLING, RATHER THAN 100% CHECKING

When controlling the quality of a batch of products, it is not practical to inspect 100% of them (unless the quantity is very small). Inspecting a large number of products takes a long time: it is expensive, and inspectors are less effective as they get tired. Actually, a 100% check does not yield that much more information than inspecting a statistically representative sample.

There is a fairly obvious principle in statistical quality control: the greater the order quantity, the higher the number of samples to check.

But should the number of samples only depend on the order quantity? What if this factory had many quality problems recently and one suspect there are many defects? In this case, one might want more products to be checked.

On the other hand, if an inspection requires tests that end up in product destruction, shouldn't the sample size be drastically reduced? And if the quality issues are always present on all the products of a given batch (for reasons inherent to processes at work), why not check only a few samples?. For these reasons, different levels are proposed by MIL-STD 105 E (the widely recognized standard for statistical quality control).

10.3 TYPES OF ACCEPTANCE SAMPLING PLANS

There are a number of different ways to classify acceptance sampling plans. One major classification is by attributes and variables. Attributes are quality characteristics that are expressed on a "go, no-go" basis. Variables are quality characteristics that are measured on a numerical scale. However, we shall restrict our self only to lots by lot acceptance sampling plans for attributes.

Another way of classification is based on the number of samples such as single sample plan, double sample plan, multi sample plan and sequential sampling. When a decision to accept or reject the lot is taken on the basis of single sample from the lot, it is known as single sampling plan. This plan operates on a lot by lot basis and is completely defined by three parameters; the lot size (N), the sample size (n) and the acceptance number (c). Single sampling plan will be in the coming section. Double sampling plan is somewhat more complicated. Following an initial sample, a decision based on the information in that sample is made to accept the lot, reject the lot or take a second sample. If the second sample is taken, the information from both the samples is combined in order to reach a decision whether to accept or reject the lot. A multistage inspection plan extension of the double sampling concepts, in that more than two samples are required in order to reach a decision regarding the disposition of the lot. The ultimate extension of multiple sampling is sequential sampling, in which units are selected from the lot one at a time, and following inspection of each unit, a decision is made either to accept the lot, reject the lot, or select another unit.

10.4 ADVANTAGES OF ACCEPTANCE SAMPLING PLANS

Checking 100% of the quantity would be long and expensive. A solution is to select samples at random and inspect them, instead of checking the whole lot. But how many samples to select? On the one hand, checking only a few pieces might prevent the inspector from noticing quality issues; on the other hand, the objective is to keep the inspection short by reducing the number of samples to check.

However, Acceptance sampling is most likely to be useful in the following situations:

(i) When testing is destructive

(ii) When the cost of 100% inspection is extremely high.

(iii) When 100% inspection is not technologically feasible.

(iv) When there are many items to be inspected and the inspection error rate is sufficiently high.

(v) When there are potentially serious product liability risks, and although the producer's process is satisfactory, a program for contingency monitoring the product is necessary.

(vi) The most effective use of acceptance sampling is not to inspect quality into the product but rather as an audit to ensure that the output of a process conforms to requirements.

10.5 BASIC CONCEPTS IN INSPECTION PLANS

Three fundamental concepts that are required in inspection activity and we should be familiar with when it comes to quality inspections:

Inspection levels

The AQL

When to inspect?

1. INSPECTION LEVELS

The relevant standards propose a standard severity, called “normal level”, which is designed to balance these two imperatives in the most efficient manner.

When to adopt a different level

Suppose you source a product from a factory that often ships substandard quality. You know that the risk is higher than average. How to increase the discriminating power of the inspection? You can opt for the level III, and more samples will be checked. Similarly, if a supplier has consistently delivered acceptable products in the past and keeps its organization unchanged, you can choose level I. As fewer samples have to be checked, the inspection might take less time and be cheaper.

The “special levels”

Inspectors frequently have to perform some special tests on the products they are checking. In some cases the tests can only be performed on very few samples, for two reasons:

They might take a long time (e.g. doing a full function test as per claims on the retail box).

They end up in product destruction. (e.g. unstitching a jacket to check the lining fabric). For these situations only, the inspector can choose a “special level”.

2. THE AQL (ACCEPTANCE QUALITY LIMIT)

The AQL is the proportion of defects allowed by the buyer. It should be communicated to the supplier in advance.

The three categories of defects

Some defects are much worse than others. Three categories are typically distinguished:

Critical defects might harm a user or cause a whole shipment to be blocked by the customs.

Major defects are not accepted by most consumers, who decide not to buy the product.

Minor defects also represent a departure from specifications, but some consumers would still buy the product.

For most consumer products, critical defects are not allowed, and the AQL for major defects and minor defects are 2.5% and 4.0% respectively.

SOME IMPORTANT REMARKS:

The number of defects is not the only cause for acceptance of refusal. The products can be refused because they are not conforming to the buyer's specifications, even though their workmanship is very good.

If you have two different products (made with different processes or in different factories), you should do two separate inspections. If you inspect them together, one product might be accepted even though it presents too many defects. Why? Because the better workmanship of the other product might "compensate" for its poor quality.

3. When to inspect?

With detailed product specifications, a QC inspector can check your products and reach a conclusion (passed or failed).four type of inspection can be done

Pre-production inspection: This type of inspection is necessary if you want to check the raw materials or components that will be used in production. Buying cheaper materials can increase a factory's margin considerably, so you should keep an eye on this risk. It can also be used to monitor the processes followed by the operators.

During production inspection: This inspection allows you to get a good idea of average product quality, and to ask for corrections if problems are found. It can take place as soon as the first finished products get off the line, but these samples might not be representative of the whole order. So usually an inspection during production is done after 10-30% of the products are finished.

Final (pre-shipment) inspection: Inspecting the goods after they are made and packed is the standard QC solution of most importers. The inspector can really check every detail,

10.6 SOME BASIC CONCEPTS SQC

The main objective of inspection is to control the quality of the product by critical examination at strategic points. Sampling inspection besides keeping down the cost of production also ensures that the quality of a lot accepted is according to the specifications. Before giving upon a detailed discussion of single and double sampling plans, we will discuss below some basic concepts which are of significant importance in their discussions.

Acceptance Quality Level (AQL)

A lot with relatively small fraction defective say p_1 that we do not wish to reject more often than a small proportion of time is sometimes referred to as a good lot. In other words, the producer wishes to maintain the standard of quality at a specified level agreed to by the consumer; this level is called Acceptance Quality Level (AQL).The AQL is a percent defective that is the base line requirement for the quality of the producer's product. The producer would like to design a sampling plan such that there is a high probability of accepting a lot that has a defect level less than or equal to the AQL.

Lot Tolerance Percentage. Defective (LTPD)

The lot tolerance percentage defective (LTPD) is the maximum fraction defective which the consumer is prepared to tolerate in an accepted lot. It is generally denoted by P_t .In other words LTPD is a designated high defect level that would be unacceptable to the consumer. The consumer would like the sampling plan to have a low probability of accepting a lot with a defect level as high as the LTPD.

Process Average Fraction Defective

It is denoted by $\overline{p}$ and it represents the quality turned out by the manufacturing process over a long period of time. In industry, the quality of any process tends to settle down to some level which may be expected to be more or less the same every day for a particular machine. If this level could be maintained and if process is working free from assignable causes of variation, the inspection could be often dispensed with. But in practice, as a result of failure of machine and men, the quality of the product may suddenly deteriorate. The process average for any manufacturing product is obtained by finding the percentage of defectives in the product over a fairly long time.

Consumer's and Producer's Risk

Any sampling scheme that involve certain risk on the part of the consumer in the sense that he has to accept certain percentage of defective lots. More precisely the probability of accepting a lot with fraction defective p_t is called consumer's risk and it is written as P_C. It is usually denoted by β. Thus

$$P\ [accepting\ a\ lot\ of\ quality\ p_t\ ] = \beta$$

Further, the producer has also to face the situation that some good lots will be rejected. He might demand adequate protection against such rejections happening too frequently just as the consumer can claim reasonable protection against accepting too many bad lots. The probability of rejecting a lot with $100\bar{p}$ as the process average percent defective is called the producers risk. It is written as Pa and usually denoted by a. Thus

$$P_p = P\ [Rejecting\ a\ lot\ of\ quality\ \bar{p}] = \alpha$$

It is termed as producer's risk.

Average Outgoing Quality (AOQ)

A common procedure when sampling and testing is non-destructive, is to 100% inspect rejected lots and replace all defectives with good units. In this case, all rejected lots are made perfect and the only defects left are those in lots that were accepted. AOQ's refer to the long term defect level for this combined LASP and 100% inspection of rejected lots process.

Sometimes the consumer is guaranteed a certain quality level after inspection; regardless of what quality level is being maintained by the producer. Let the producer's fraction defective be p. This is termed as incoming quality. The expected fraction defective remaining in the lot after the sampling inspection plans is termed as Average Outgoing Quality (AOQ). It is denoted by $\tilde{p}$. Obviously it is a function of the incoming quality p. The A.O.Q. values are given by the following relation

$$A.O.Q. = \tilde{p}.\frac{(N - n)P_a}{N}$$

Where N is lot size, ii is sample size and Pa is the probability of acceptance of the lot. In this formula, it is assumed that all defective items found are repaired or replaced by good items. If n is small as composed to N, then the A.O.Q. is given by

$A.O.Q. = \tilde{p}.P_a$ Where N is the lot size.

Average Outgoing Quality Limit (A.O.Q.L.)

The maximum value of p subject to variations in p is called the average out going quality limit. It is denoted by $\bar{p}_L$. If p_M is the value of p which maximize, $\tilde{p}$, then

$\bar{p}_L = AOQL =$ ATI = n+ (1 - pa) (N – n)

where N is the lot size.

Operating Characteristics Curve (O.C. Curve)

OC Curves or Operating Characteristic Curves refer to a graph of attributes of a sampling plan considered during management of a project which depicts the percent of lots or batches which are expected to be acceptable under the specified sampling plan and for a specified process quality.

The specified sampling plan may be singular, sequential or iterative and may be using a particular size of a sample depending upon the demands of the project and could yield the results of acceptance or rejection based on specified criteria.

It is a graphic representation of the relationship between the probability of acceptance a for variations in the lot quality p. It is the most important curve as it also represents the two risks. An O. C. curve is said to be ideal if probability of acceptance of a lot having proportion defectives p or less is equal to unity and the probability of rejecting a lot with proportion defective greater than p is also equal to unity.

OC Curve Uses

* It helps in the selection of sampling plans

* It aids in the selection of plans that are effective in reducing risks.

* It can help in keeping the high cost of inspection low.

Average Total Inspection (ATI): When rejected lots are 100% inspected, it is easy to calculate the ATI if lots come consistently with a defect level of p. For a LASP (n,c) with a probability pa of accepting a lot with defect level p, we have

ATI = n + (1- pa) (N - n)

where N is the lot size.

Average Sample Number (ASN): For a single sampling (n,c) we know each and every lot has a sample of size n taken and inspected or tested. For double, multiple and sequential, the amount of sampling varies depending on the number of defects observed. For any given double, multiple or sequential plan, a long term ASN can be calculated assuming all lots come in with a defect level of p. A plot of the ASN, versus the incoming defect level p, describes the sampling efficiency of a given Lot Acceptance Sampling Plan

It is the expected sample size required for coming to a decision about accepting or rejecting the lot under the sampling plan. Naturally it is function of the incoming quality p. A.S.N. curve exhibits the relationship between incoming fraction defective and the average total number inspected per lot. The ASN is given by

$$ASN = n + (1 - P_a)(N - n)$$

$$= nP_a + N(1 - P_a)$$

Where P_a is the probability of acceptance of the lot of quality level p.

10.7 SUMMARY AND FURTHER SUGGESTED READNG

In this lesson we have explained acceptance sampling, its meaning, aspects of sampling, and its uses. The different types of sampling plan such as single sample plan, double sampling plan, multi-sampling plan and sequential sampling have also been explained. However, these sampling plans shall be discussed in detail in the forthcoming lessons. Some basic concepts, which are widely used in sampling plans, have also been discussed here such as AQL, LTPD, AOQ, A.S.N ,A.T.I,AOQL consumers and producers risks.

FURTHER SUGGESTED READINGS

1. D.C. Montgomery Introduction to Statistical Quality Control.

10.6 SELF ASSESSMENT QUESTIONS

1. What do you understand by principle of acceptance sampling how it is preferred over 100% inspections?

2. Explain the terms AQL, LTPD, AOQ, A.T.I, A.O.Q.L, consumers and producers risks.

3. What do you understand by the terms average sample number and average amount of total inspection in context to the quality of a product.

4. What do you understand by acceptance sampling procedure? State its uses giving illustrations.

11.1 OBJECTIVES

The major objectives of this lesson are

- To introduce the students with the working and general procedure of single sampling plan
- To give the basic knowledge about the behavior of single sampling plan in the terms of its OC function, ASN and ATI function of this sampling scheme.

SINGLE SAMPLING PLAN AN INTRODUCTION

One sample of items is selected at random from a lot and the disposition of the lot is determined from the resulting information. These plans are usually denoted as (n,c) plans for a sample size n, where the lot is rejected if there are more than c defectives. These are the most common (and easiest) plans to use although not the most efficient in terms of average number of samples needed

Here decision about accepting or rejecting a lot is taken on the basis of single sample only; then the acceptance plan is known as single sample plan. This plan operates on a lot by lot basis and is completely defined by following three numbers

i) the lot size N,

(ii) the sample size n, and

(iii) the acceptance number c, usually the c would be the maximum allowable number of defective items for acceptance.

11.2 WORKING OF SINGLE SAMPLING PLAN (GENERAL PROCEDURE)

Single sampling plan:. One sample of items is selected at random from a lot and the disposition of the lot is determined from the resulting information. These plans are usually denoted as (n,c) plans for a sample size n, where the lot is rejected if there are more than c defectives. These are the most common (and easiest) plans to use although not the most efficient in terms of average number of samples needed.

Let N be the lot size; n, the sample size; c, the acceptance number, i.e. maximum allowable number of defectives in the sample. The single sample plan may be described as follows

(i) Select a random sample of size n from a lot of size N.

(ii) Each article in the sample is inspected and then classified as defected or nondefective. Let d be the number of defectives in the sample.

(iii) If the number of defective articles in the sample is less than or equal to c

i.e. $d \leq c$, then accept the lot, and replace all defective found in the sample by non-defectives.

(iv) If the number of defectives is greater than c, i.e. d> c, reject the lot. Inspect the remaining lot and replace all defectives found by non-defectives.

In this sampling plan, the chance of cent-per-cent inspection increases as the percentage of defectives in the lot increases.

For example, if the lot size N is 2000 sample size n = 150 and c =2, then the sampling plan means that from the lot of 2000 items a random sample of n = 150 units is inspected and the number of defective items d observed. If the number of observed defectives d is greater than c, the lot will be rejected and if the observed defectives d is less than or equal to c , the lot will be accepted. This type of procedure is called a single sampling plan because the lot is sentenced on the base on the information contained in one sample of size n.

11.3 DETERMINATION OF N AND C IN SINGLE SAMPLING PLAN

The two quantities n and c may be determined by two different approaches:

i. **Lot Quality Protection:** In this approach the values of n and c are determined from the pecified values of N, p_t, $\overline{p}$ and P_c, where

p_t = lot tolerance proportion defectives,

$\overline{p}$ =producers process average fraction defective

Pc = consumer's risk

ii. **Average Quality Protection:** In this approach, the problem of protecting the consumer from an inferior product is solved by ensuring him a certain quality level of the product after inspection regardless of what quality level is being maintained by the producers.

Suppose in a lot of size N, incoming quality is p, then the number of defective items is Np and non- defective items is N—Np = N(l—p). The probability of getting exactly x defectives in a sample of n from this lot is given by

$$f(x, p) = \binom{N_p}{x}\binom{N - N_p}{n - x} \bigg/ \binom{N}{n}$$

Therefore, the probability of accepting a lot of quality p is

$$\sum_{x=0}^{c} f(x, p) = \sum_{x=0}^{c} \binom{N_p}{x}\binom{N - N_p}{n - x} \bigg/ \binom{N}{n} \qquad \ldots\ldots\ldots\ldots..(1)$$

Hence the consumer's risk is given by

$$P_c = P\,(Accepting\ a\ lot\ of\ quality\ p_t\,)$$

$$= \sum_{x=0}^{c} f(x, p_t)$$

$$= \sum_{x=0}^{c} \binom{N_p}{x}\binom{N - N_{p_t}}{n - x} \bigg/ \binom{N}{n} \qquad \ldots\ldots\ldots\ldots..(2)$$

To protect himself against poor quality, the consumer usually, demands a small value of P_C for given p_t

The producer's Risk is given by

$$P_p = P\{Rejecting\ a\ lot\ of\ quality\ \overline{p}\ \}$$

$$= 1 - P\{Accepting\ a\ lot\ of\ quality\ \overline{p}\}$$

$$= 1 - \sum_{x=0}^{c} f(x, \overline{p})$$

$$= 1 - \sum_{x=0}^{c} \binom{N_{\overline{p}}}{x}\binom{N - N_{\overline{p}}}{n - x} \Big/ \binom{N}{n} \qquad \ldots\ \ldots\ldots(3)$$

If the process average fraction defective is as claimed by the producer then the total average amount of inspection per lot is

$$I = n + (N - n)P_p \qquad \ldots\ldots.. (4)$$

Since n items have to be inspected in each case and the remaining (N—n) items will be inspected only if d> c.

The computation of probabilities given by (1) and (3) are extremely difficult, thus on using binomial approximation to hyper-geometric distribution, the consumer's and producer's risk becomes

$$P_c = \sum_{x=0}^{c} \frac{(Np_t)!}{(Np_t - x)!}\left(\frac{n}{N}\right)^n \left(1 - \frac{n}{N}\right)^{Np_{t-x}} \qquad \ldots\ldots\ldots\ldots(5)$$

And

$$P = 1 - \sum_{x=0}^{c} \frac{(N)!}{(N - x)!}(\overline{p})^x (1 - \overline{p})^{n-x} \qquad \ldots\ldots\ldots.. (6)$$

In most of the practical problems, $\overline{p}$ is likely to be less than 0.10 and n islikely to be sufficiently large to warrant the use of Poisson distribution as a approximation to binomial distribution. Therefore, producer's risk can further be approximated by

$$P_p = 1 - \sum_{x=0}^{c} (n\bar{p})^2 \frac{e^{-n\bar{p}}}{x!} \qquad \text{..........(7)}$$

Consumer's requirement fixes the values of P_c and p_t. N is always fixed. For given values of P_c and p_t, the equation (2) which in values two unknown n and c is satisfied by a large number of pairs of n and c. To safeguard producer's interest also, out of these possible pairs one involving the minimum amount of inspection as given in (4) is chosen. Though theoretical computations are quite difficult and time consuming, Dodge and Roming, by applying numerical methods of solutions of equations, have prepared extensive tables for minimizing values of n and c for $P_c = 0.10$ and different values of .

Some important definition in context with single sampling plan.

The following are the some important measures which tell us how a given single sampling plan behaves on lots of various fraction defectives:

11.4 THE O.C CURVE OF SINGLE SAMPLING PLAN

O.C. Curve:

An important measures of the performance of an acceptance sampling plan is the operating-characteristic (OC) curve. This curve plots the probability of accepting the lot versus the lot fraction defective. Thus, the O.C. curve displays the discriminatory power of the sampling plan. That is, it shows the probability that a lot submitted with a certain fraction defective will be either accepted or rejected.

The O.C. curve for the incoming quality 'p' is given by

$$P_p = L(p) = \sum_{x=0}^{c} f(x, p)$$

$$= \sum_{x=0}^{c} \binom{N_p}{x} \binom{N - N_p}{n - x} \Big/ \binom{N}{n}$$

If p <0.10, a good approximation to L (p) is given by

$$L(p) \cong \sum_{x=0}^{c} \binom{N_p}{x} \left(\frac{n}{N}\right)^x \left(1-\frac{n}{N}\right)^{N_p - x}$$

Further if p < 0.10 and also $\frac{n}{N} < 0.10$, then

$$L(p) \cong \sum_{x=0}^{c} \binom{N_p}{x} \left(\frac{n}{N}\right)^x \left(1-\frac{n}{N}\right)^{N_p - x}$$

$$L(p) \approx \sum_{x=0}^{c} \frac{e^{-np}(np)^p}{x!}$$

11.5 A.O.Q AND A.O.Q.L

A.O.Q AND AOQL : If p is the incoming quality, there will be no defectives left in a lot of six N if the sample contains more than c defectives i.e., if $x > c$ .On the other hand, if the sample contains less than or equal to c defectives, then the number of defectives in a lot of size N is $Np - x$. Thus the mean value of the number of defectives after inspection is given by

$$m = 1 - \sum_{x=0}^{c} \left(N_p - x\right) f(x, p) + \sum_{x=c+1}^{c} 0.\, f(x, p)$$

$$= \sum_{x=0}^{c} \left(N_p - x\right) \binom{N_p}{x} \binom{N - N_p}{n - x} \Big/ \binom{N}{n}$$

$$= N \sum_{x=0}^{c} \left(p - \frac{x}{N}\right) \binom{N_p}{x} \binom{N - N_p}{n - x} \Big/ \binom{N}{n}$$

Therefore, the mean value of the fraction defective after inspection (A.O.Q) will be

$$A.O.Q. = \tilde{p} = \frac{m}{n}$$

$$= \sum_{x=0}^{c} \left(p - \frac{x}{N} \right) \binom{N_p}{x} \binom{N - N_p}{n - x} \bigg/ \binom{N}{n}$$

The maximum value of $\tilde{p}$ subject to variations in p is called the AOQL which is denoted by $\tilde{p}_L$ .The AOQL takes care of consumer's interest. Dodge and Roming have prepared extensive AOQL tables for minimizing values of n and c.

11.6 A.T.I AND A.S.N OF SINGLE SAMPLING PLAN

Average Total Inspection (A.T.I.)

The other concept that is useful in consideration of a single sampling plan is the Average Total Inspection (A.T.L). The A.T.L can be calculated through the following steps for a single sampling plan:

(i) If the lot is accepted then the inspection amount is only ii units.

(ii) If the lot is rejected, then the amount of inspection is N.

Further, the probability of acceptance of a lot is P_a and rejection is $1 - P_a$ therefore, the Average Total Inspection (A.T.I.) is given by

$$A.T.I. = n\,P_a + N(1 - P_a)$$

$$= N + (n - N)P_a$$

$$= n + (N - n)(1 - P_a)$$

By taking the values of various A.T.I.'s for different values of P_a at Y axis and values of p at X-axis, we obtain an A.T.1 curve.

Average Sample Number Curve (ASN Curve): In a single sampling plan, the minimum number of articles inspected in n, the sample size. The remaining $N - n$ articles are inspected only when the lot is rejected with a probability P_p . If I denote the average number of articles inspected per lot under single sampling plan, then it is given by

$$I = n + (N - n)P_c$$

$$= n + (N - n)\left[1 - \sum_{x=0}^{c} (n\bar{p})^2 \frac{e^{-n\bar{p}}}{x!}\right]$$

On plotting the ASN function we obtain the ASN curve

ILLUSTRATION: Suppose you are given a lot containing of 1000 items and a sample of 100 item is taken. If it contains 2 or less defective items, the lot is accepted otherwise rejected. Plot the P.C., ASN and ATI curve.

Solution: Here we have given a single sampling plan characterized by

$N = 1000,\ n = 100$ And $c = 2$.

Since we have $\frac{n}{N} = \frac{100}{1000} = 0.10 \leq 0.10$ and for $p < 0.10$, the probability of acceptance L_p or P_a is given by

$$P_a = \sum_{x=0}^{c} \frac{e^{-np}(np)^x}{x!} = \sum_{x=0}^{2} \frac{e^{-\lambda}(\lambda)^x}{x!}$$

where $\lambda = np$ and p is the submitted lot quality.

The probabilities, given by (1), for different values of λ. and c = 2 for p from 0 to 1, are given in the table below

P	$np = \lambda$	P_a	$p.P_a$ (AOQ)	nP_a	$1 - P_a$	$N(1 - P_a)$	$nP_a + N(1 - P_a)$ (ATI)
0.01	1	0.920	0.0092	92.00	0.08	80.00	172 00
0.02	2	0.677	0.0135	67.70	0.323	323.00	390.70
0.03	3	0.423	0.0127	42.30	0.577	577.00.	619.30
0.04	4	0.238	0.0095	23.80	0.762	762.00	785.80
0.05	5	0.125	0.0063	12.50	0.875	875.00	887.50
0.06	6	0.062	0.0037	6.20	0.938	938.00	944.20
0.07	7	0.030	0.0021	3.0	0.970	970.00	973.00
0.08	8	0.014	0.0011	1.40	0.986	986.00	987.40
0.09	9	0.006	0.0005	0.60	0.994	994.00	994.60
0.10	10	0.003	0.0003	0.30	0.997	997.00	997.30

The Average Total amount that is inspected under the above sampling plan is given by

$$A.T.I. = n\,P_a + N(1 - P_a)$$

$$And,\ A.O.Q = p.P_a$$

Both A.T.I. and A.O.Q are also presented in the table given above. The O.C. curve is plotted in figure. 3.3.1, A.TI curve is presented in figure 3.3.2 and A.O.Q. curve is presented in figure 3.3.3.

Figure, 3.3.1

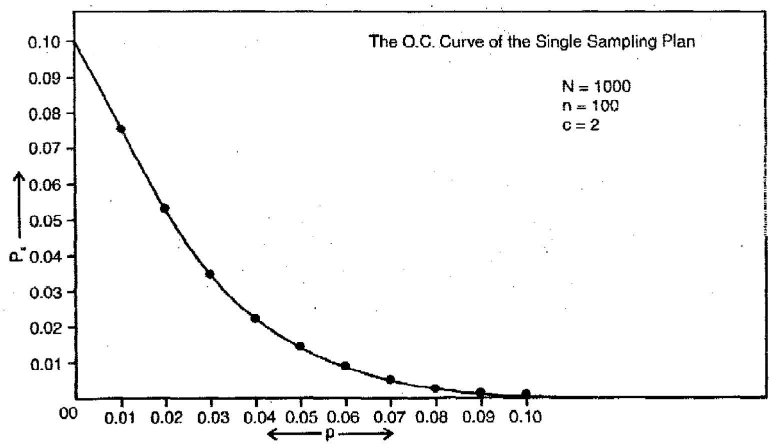


Figure 3.3.2

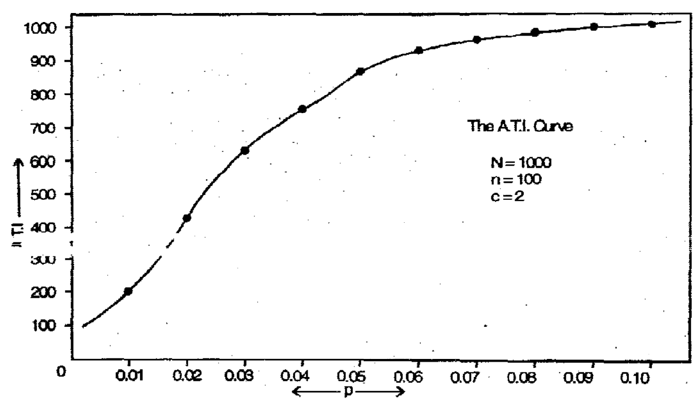
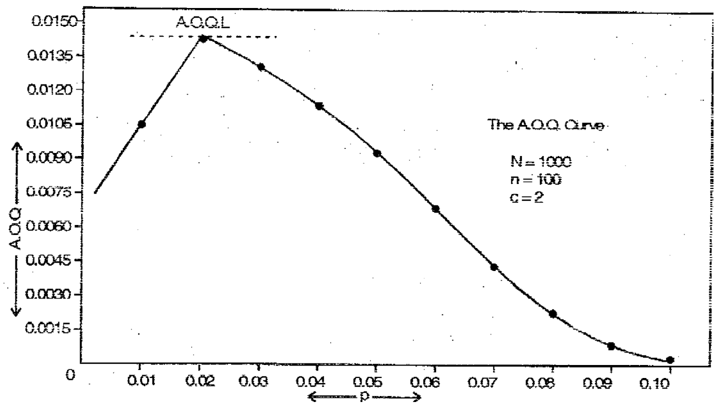


figure 3.3.3



11.7 SELF ASSESSMENT QUESTIONS

1. Describe the Single Sampling Plan.

2. How would you determine the values of n and c in single sampling plan?

3. Draw O.C. curve for single sampling plan with N = 2250, n=225 and c=12.

4. Plot A,T.1 and AO.Q curves for single sampling plan with N =10000, n=200 and c=4

5. Describe single sampling plan. Obtain OC and AOQ curve for this plan. Distinguish clearly between: (i) Producer's risk and Consumer's risk ; (ii) AQL and LTPD.

6. Explain the principles and the procedures of (i) Lot Quality Protection, and (ii) Average Quality Protection, assured to consumers by sampling inspection plans.

7. Describe the single sampling plan for acceptance sampling, deriving expressions for the producer's and consumer's risks, and show that approximately.

$$\text{ATI} = n + (N - n)\left[1 - \sum_{x=0}^{c} (n\bar{p})^{x} \frac{e^{-n\bar{p}}}{x!}\right]$$

11.8 SUMMARY AND FURTHER SUGGESTED READING

The major objectives of this lesson were to introduce the students with the working and general procedure of single sampling plan and to give the basic knowledge about the behavior of single sampling plan in the terms of its OC function, of this sampling scheme. We have learnt about the concepts that are useful in consideration of a single sampling plan is the Average Total Inspection (A.T.I) .We have discussed the techniques to determine the values of n and c in the single sampling plan and how to plat the ATI and AOQ curves of single sampling plan.

12.1 OBJECTIVES

After successful completion of this lesson, the students will be able to:

- Know the meaning of double sampling plan,
- Distinguish between single and double sampling plan,
- Understand the need of double sampling plan
- Determine the O.C. function of double sampling plan
- Obtain consumer's and producer's risks, and
- Determine the A.S.N. curve for double sampling plan.

12.2 DOUBLE SAMPLING PLAN AN INTRODUCTION

This sampling scheme is propounded by Dodge and Roming known as double sampling plan. A double sampling is a procedure in which, under certain circumstances, a second sample is required, if the first sample fails, to accept or reject the lot Double sampling plans: After the first sample is tested, there are three possibilities:

1 Accept the lot　　　2 Reject the lot　　　3 No decision

If the outcome is (3), and a second sample is taken, the procedure is to combine the results of both samples and make a final decision based on that information. A double sampling plan is described by the following parameters:

$n_1 = sample\ size\ on\ the\ first\ sample$

$c_1 = acceptance\ number\ of\ the\ first\ sample$

$n_2 = sample\ size\ on\ the\ second\ sample$

$c_2 = acceptance\ number\ for\ both\ samples.$

$d_1 = number\ of\ defective\ items\ in\ the\ first\ sample,$

$d_2 = number\ of\ defective\ items\ in\ the\ second\ sample.$

4.3 Working of double PLAN (general procedure)

The general procedure of double sampling plan is described in the following steps:

Procedure

(i) Take a sample of size n, from the lot.

(ii) If $d_1 \leq c_1$, accept the lot and replaced the defective, found in the sample by good items.

(iii) If $d_1 > c_2$, reject the whole lot.

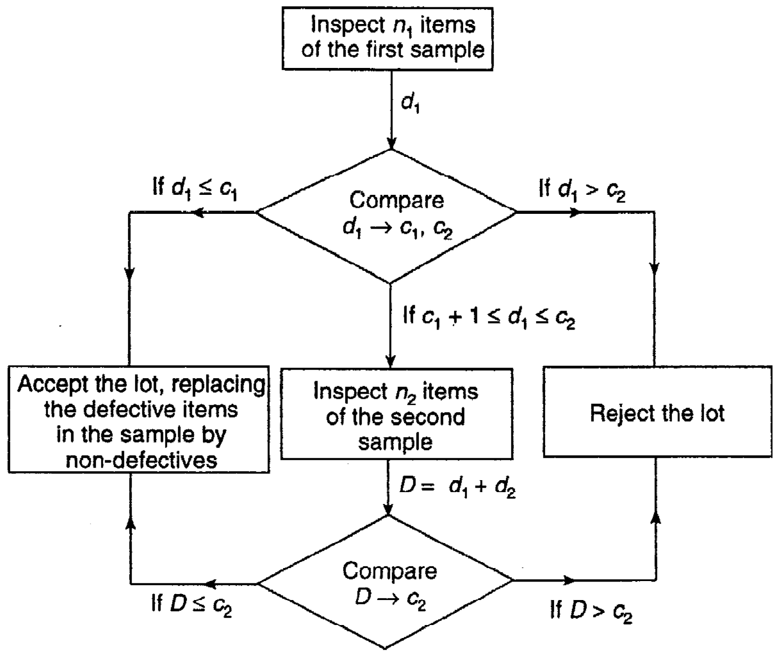
(iv) If $c_1 < d_1 \leq c_2$, take a second random sample of size n_2 from the remaining lot.

(v) If $d_1 + d_2 \leq c_2$, accept the lot and replaced all the defective items with good items.

(vi) If $d_1 + d_2 > c_2$, reject the lot.

For example, suppose $n_1 = 50, c_1 = 2, n_2 = l00, c_2 = 3$ and N =1000. Thus, a random sample of size $n_1 = 50$ items is taken from the lot, and the number of defectives in the sample, d_1, is observed. If $d < c_1 = 2$, the lot is accepted on the first sample. If $d_1 > c_2$, the lot is rejected on the first sample. If $c_1 < d_1 \leq c_2$, a second random sample of size $n_2 = l\,00$ is drawn from the lot, and the number of defectives in this second sample, d_2, is observed. Now the combined number of observed defectives from both the samples, $d_1 + d_2$, is used to determine to accept or reject the lot. If $d_1 + d_2 \leq c_2 = 3$, the lot is accepted. However, if $d_1 + d_2 > c_2 = 3$, the lot is rejected.

FLOW CHART FOR DOUBLE SAMPLING PLAN PROCEDURE



12.4 DOUBLE SAMPLING PLAN VERSUS SINGLE SAMPLING PLAN

Making a final choice between single or multiple sampling plans that have acceptable properties is a matter of deciding whether the average sampling savings gained by the various multiple sampling plans justifies the additional complexity of these plans and the uncertainty of not knowing how much sampling and will be done on a day-by-day basis.

The principal advantage of a double sampling plan with respect to single sampling is that it may reduce the total amount of required inspection and thus the cost of inspection will be lower for double sampling than it would be for single sampling plan. It is also possible to reject without complete inspection of the second sample.

Furthermore, in some situations, the use of double sampling plan has the psychological advantage of giving a lot a second chance. This may have some appeal to the vendor. However, there is no real advantage to double sampling in this regard, because single and double sampling plans can be chosen so that they have same O.C.curves. Thus both plans would offer the same risks of accepting or rejecting lots of specified quality

Double sampling has two potential disadvantages. First, unless curtailment is used on the second sample, under some circumstances double sampling may require more total inspection than would be required in a single sample plan that offers the same protection. Thus, unless double sampling is used carefully, its potential economic advantage may be cost. The second disadvantage of double sampling plan is that it is administratively more complex, which may increase the opportunity for the Occurrence of inspection errors. Furthermore, there may be problems in storing and handling raw materials or components parts for which single sample has been taken, but that are awaiting a second sample before a final lot dis positioning decision can be made. The general reduction in the amount of inspection afforded by double sampling is one of its strongest advantages. This does not necessarily mean, however, that a double sampling scheme could be less costlier than the single sampling scheme. The double sampling schemes being more complicated and the necessity of inspecting second sample being unpredictable, the unit cost of inspection for a double sampling procedure may be higher than that for single sampling procedure.

12.5 THE O.C CURVE OF DOUBLE SAMPLING PLAN

The performance of a double sampling plan can be conveniently summarized by means of it operating characteristic (O.C.) curve. The O.C. curve for double sampling plan is somewhat more involved than the O.C. curve for single sampling plan. A double sampling plan has a primary O.C. curve that gives the probability of acceptance as a function of lot or process quality. It also has supplementary O.C. curves that show the probability of lot acceptance or rejection on the first sample. The O.C. curve for the probability of rejection on the first sample is simply the O.C.curve for the single sampling plan $n = n_1$ *and* $c = c_2$.

The lot will be accepted under the following mutually exclusive ways:

(ii) If $0 \le d_1 \le c_1$, $\quad (ii) \quad d_1 = c_1 + 1, \quad d_2 = c_2 - c_1 - 1$

$(iii) \quad d_1 = c_1 + 2; \quad d_2 = c_2 - c_1 - 2$

$$\begin{array}{cc} \vdots & \vdots \\ d_1 = c_2 & d_2 = 0 \end{array}$$

Hence, the probability of acceptance for a lot of incoming quality 'p' is given by

$$p_p(p) = \sum_{x=0}^{c_1} f(x,p) + \sum_{y=0}^{c_2-x}, \sum_{x=c_1+1}^{c_2} f(x,p).h[y.p/x]$$

Where f(x, p) is the probability of finding x defectives in the first sample and $h[y.p/x]$ is the conditional probability of finding y defectives in the second sample given that x defectives have already found the first sample. Thus

$$f(x,p) = \binom{N_p}{x}\binom{N-N_p}{n_1-x}\binom{N}{n_1}$$

and

$$f(x,p) = \binom{N_p - x}{y}\binom{N-n_1-N_{p+x}}{n_2-y}\bigg/\binom{N-n_1}{n_2}$$

Therefore, we get

$$P_a(p) = \sum_{x=0}^{c_1}\binom{N_p}{x}\binom{N-N_p}{n_1-x}\binom{N}{n_1}$$

$$+\sum_{y=0}^{c_2-x}, \sum_{x=c_1+1}^{c_2} \frac{\binom{N_p}{x}\binom{N-N_p}{n_1-x}\binom{N_p-x}{y}\binom{N-n_1-N_p+x}{n_2-y}}{\binom{N}{n_1}\binom{N-n_1}{n_2}}$$

$$= P_{a_1} + P_{a_2} \qquad \text{(Say)} \qquad \ldots\ldots\ldots\ldots..(1)$$

where P_{a_1} *and* P_{a_2} are the probabilities of acceptance on the first second sample respectively.

CONSUMER'S AND PRODUCER'S RISKS

The consumer's risk for double sampling plan is given by

$P_c = P\ [accepting\ a\ lot\ of\ quality\ p_t\ ]$

$$= P_a(p_t) \qquad \ldots\ldots\ldots\ldots(2)$$

and producer's risk is given by

$P_c = 1 - P\ [accepting\ a\ lot\ of\ quality\ \overline{p}]$

$$= 1 - P_a(\overline{p}) \qquad \ldots\ldots\ldots\ldots(3)$$

Thus, on replacing p with p_t and $\overline{p}$ in (1), we get the consumer's and producer's risk respectively as given below

$$P_a(p_t) = \sum_{x=0}^{c_1} \binom{N_{p_t}}{x}\binom{N - N_{p_t}}{n_1 - x}\binom{N}{n_1}$$

$$+ \sum_{y=0}^{c_2 - x}, \sum_{x=c_1+1}^{c_2} \frac{\binom{N_{p_t}}{x}\binom{N - N_{p_t}}{n_1 - x}\binom{N_{p_t} - x}{y}\binom{N - n_1 - N_{p_t} + x}{n_2 - y}}{\binom{N}{n_1}\binom{N - n_1}{n_2}}$$

And producer's risk

$$P_a(\overline{p}) = \sum_{x=0}^{c_1} \binom{N_{\overline{p}}}{x}\binom{N - N_{\overline{p}}}{n_1 - x}\binom{N}{n_1}$$

$$+ \sum_{y=0}^{c_2 - x}, \sum_{x=c_1+1}^{c_2} \frac{\binom{N_{\overline{p}}}{x}\binom{N - N_{\overline{p}}}{n_1 - x}\binom{N_{\overline{p}} - x}{y}\binom{N - n_1 - N_{\overline{p}} + x}{n_2 - y}}{\binom{N}{n_1}\binom{N - n_1}{n_2}}$$

12.6 A.T.I AND A.S.N OF DOUBLE SAMPLING PLAN

The average sample number curve of a double sampling plan is also usually of interest to the quality engineer. In single sampling plan, the size of sample inspected from the lot is always constant, whereas in double sampling, the size of the sample selected depends on whether or not the second sample is necessary. The probability of drawing a second sample varies with the fraction defective in the incoming lot. With complete inspection of the second sample, the average sample in double sampling is equal to the size of first sample times the probability that there will only be one sample, plus the size of the combined samples times the probability that a second sample will be necessary.

In short, in an acceptance-rejection double sampling plan, the number of items inspected for a lot is either n_1, (when the lot is accepted or rejected on the basis of the first sample or $(n_1 + n_2)$ when a second sample of size n_2 is drawn. Thus the expected sample size for a decision is given by:

$$\begin{aligned} ASN &= n_1 P_1 + (n_1 + n_2)(1 - P_1) \\ &= n_1 + n_2(1 - P_1) \end{aligned}$$

where P_1 is the probability of a decision (acceptance or rejection of the lot) on the basis of the first sample.

In a double sampling acceptance-rectification scheme in which rejected lots are inspected 100 percent, the average total inspection (ATI) per lot is given by:

$$1(p) = n_1 P_{a_1} + (n_1 + n_2)P_{a_2} + N(1 - P_a)$$

Since

(i) only n_1 items will be inspected if the lot is accepted on the basis of the first sample and its probability is $P_{a1}(p)$,

(ii) $(n_1 + n_2)$ items will be inspected if the lot is accepted on the basis of the second sample, and its probability is $P_{a2}(p)$, and

(iii) The entire lot of N items will be inspected if the lot is rejected and the probability of this is $1 - P_a(p)$.

Since

$$P_a = P_{a_1} + P_{a_2}$$

$$\Rightarrow \quad P_{a_2} = P_a - P_{a_1}$$

Thus

$$1(p) = n_1 P_{a_1} + (n_1 + n_2)(P_a - P_{a_1}) + N(1 - P_a)$$

$$= n_1 P_{a_1} + (n_1 + n_2)[(1 - P_{a_1}) - (1 - P_a)] + N(1 - P_a)$$

$$= n_1 P_{a_1} + n_1(1 - P_{a_1}) + n_2(1 - P_{a_1}) + [N - (n_1 + n_2)](1 - P_a)$$

$$= n_1 + n_2(1 - P_{a_1}) + [N - n_1 - n_2](1 - P_a)$$

In Dodge and Romig tables, n2 has no fixed relation to a1 but it is determined so that ATI is minimum and so that the probability of acceptance on the basis of first sample is approximately the same as the probability of acceptance on the basis of second sample.

12.7 SELF ASSESSMENT QUESTIONS

1. Describe the double sampling plan and explain the principles that form the basis of this plan.
2. Distinguish between single sampling plan and double sampling plan.
3. Define O.C. curve and A.S.N. curve for double sampling plan.
4. Define consumer's and producer's risks for double sampling plan.
5. What are single sampling plan and double sampling plan? Discuss the relative merits and demerits of single and double sampling plans.
6. What is Average Sample Number (ASN) and Average Total Inspection (ATI). Explain the method of their calculation for single sampling plan. Why are ASN and ATI calculated?
7. Explain clearly how one is led to AOQL, explaining the various intermediary concepts of acceptance sampling.
8. Explain the concepts of producer's and consumer's risk in sampling inspection schemes. Define the average sample number and the average outgoing quality in the case of Double Sampling Inspection and indicate their usefulness in choosing a sampling scheme.

12 8 SUMMARY AND FURTHER SUGGESTED READING

In this lesson, double sampling plan have been described along with its O.C curve and A.S.N curve. A double sampling plan is a procedure in which, under certain circumstances, a second sample is required to accept or reject a lot. A double sampling plan is defined by five parameters; lot size N, sample size of the first sample n_1, accept number of the first sample c_1, sample size of the second sample n_2 and acceptance number of the second sample c_2.

If number of defective items in the first and second sample are d_1 and d_2 respectively, then the procedure for double sampling plan is

(i) Take a sample of size n, from the lot.

(ii) If $d_1 \leq c_1$, accept the lot and replaced the defective, found in the sample by good items.

(iii) If $d_1 > c_2$, reject the whole lot.

(iv) If $c_1 < d_1 \leq c_2$, take a second random sample of size n_2 from the remaining lot.

(v) If $d_1 + d_2 \leq c_2$, accept the lot and replaced all the defective items with good items.

(vi) If $d_1 + d_2 > c_2$, reject the lot.

FURTHER SUGGESTED READING

1. D.C. Montgomery Introduction to Statistical Quality Control.

2. S.C Gupta and V.K Kapoor :Fundamentals of applied Statistics

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13.1 OBJECTIVES

The following are the main objectives of this lesson:

- To provide the concept of making good lots.
- To discuss the problems arising from re-submission of rejected lots,
- To introduce the concept of rectifying inspection plan, and
- To discuss the Indian Standard Tables and their applications.

13.2 AN INTRODUCTION TO RECTIFYING INSPECTION PLANS

In previous lessons we have explained the concept of consumer's and producer's risks, O.C. and A.S.N. curves, AQL, LTPD, AOQL and average amount of inspection. The various sampling plans have also been discussed in detail such as single sampling plan and double sampling plan. In the following sections we shall discuss lot by lot Sampling plans in which a specified quality objective is attained through corrective inspection of rejected lot .The inspection of the rejected lots and replacing the defective pieces found in the rejected lots by the good ones, eliminates the number of defectives in the lot to a great extent, thus improving the lot quality. These plans are called 'Rectifying Inspection Plans' also we are going to discuss the Indian Standard Table.

13.3 THE FORMULATION OF INSPECTION LOTS

In accepting sampling inspection, an inspection lot is a group of articles accepted or rejected on the basis of one or more samples. How the lot is formed can influence the effectiveness of the acceptance sampling plan.

An inspection lot is not necessarily identical with a production lot, a purchase lot, or a lot for other purposes. Many practical matters such as rate of production and availability of storage space necessarily influence the formation of lots. There are a number of important considerations in forming lots for inspection. Some of these are as follows

1. Lots should be homogeneous. The units in the lot should be produced by the same machines, same operators, and from common raw material, at approximately the same time. When lots are non-homogeneous, such as when the output of two different production lines is mixed, the acceptance sampling plan may not function as effectively as it could. Non-homogeneous lots also make it more difficult to take corrective action to eliminate the source of defective products.

2. Larger lots are preferred over smaller lots. It is usually more economically efficient to inspect large lots than small lots.

3. Lots should be conformable to the materials handling systems used in both vendor and consumer facilities. In addition, the items in the lots should be packaged so as to minimize shipping and handling risks, and so as to make selection of the units in the samples relatively easy.

   The desire to have each lot come from a homogeneous source obviously conflicts with the desire to have large lots. Practical decisions usually call for a compromise between these two objectives.

13.4 EFFECTS OF RESUBMISSION OF REJECTED LOTS

Problems Arising From Resubmission of Rejected Lots: Whenever rejected lots are returned to the producers, the resubmission of these lots for another sampling inspection creates certain problems. Obviously it is not in the consumer's interest for lots to be submitted unchanged Suppose that the probability of acceptance of a lot of given quality is 0.80. This will be the probability of its acceptance not only the first submission but also on any subsequent submission. The probability of acceptance with a maximum of two re- submission is as follows:

Probability of acceptance on first submission = 0.80

Probability of rejection on first submission followed by acceptance on second submission = (1-0.80)(0.80)= 0.16

Probability of rejection on first and second submission followed by acceptance on third submission. = (1-0.80)(1-0.80)(0.80)= 0.03

Probability of acceptance with not more three submissions = 0.80+0.16+0.03 = 0.99

Here the normal action by a rutable producer is to screen a rejected lot before resubmitting it. Thus, lots found unacceptable shall be resubmitted for re inspection only after all units are re-examined and all defective units are removed. Re-submitted lots that have received hundred percent inspections by the producer after rejection presumable are considerably better on their second submission than they were on the first submission. Therefore, from the sampling inspection of resubmitted lots should not be viewed as representative of their process average. Criterion for determining whether normal, tightened or reduced inspection is to be used should be based solely on the results of original inspection, and the results from any re-submission lots should be avoided.

13.5 WORKING OF RECTIFYING INSPECTION PLANS

Accepting sampling plans usually require corrective action when lots are rejected. This generally takes the form of 100% inspection or screening of rejected lots, with all discovered defective either removed for subsequent rework or return to the producer. Such sampling programs are called rectifying inspection programs, because the inspection activity affects the final quality of the outgoing product.

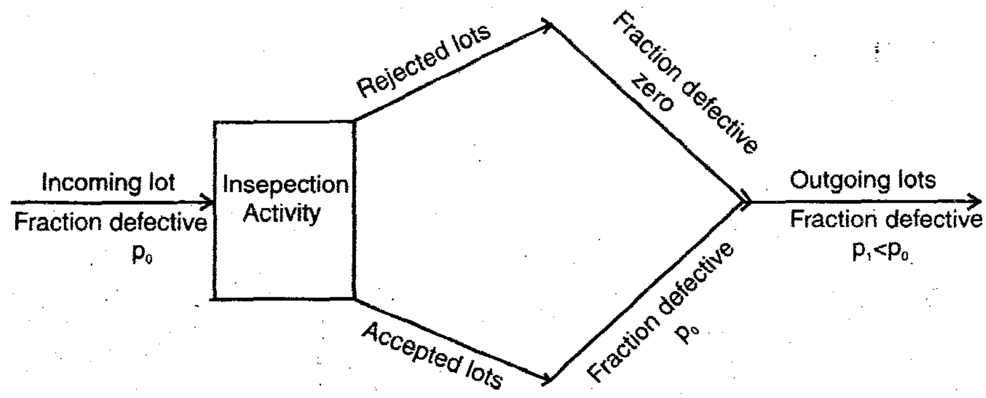
These plans enable the manufacturer to have an idea about the average quality of the product that likely to result at a given stage of manufacture through the combination of production, sampling inspection and rectification of rejected lots.

Most of the rectifying inspection plans for lot by lot sampling call for 100% inspection of the rejected lots and replacing the defective pieces found by good ones. The two important points related to rectifying inspection plans are

(i) The average quality of the product after sampling and 100% inspection of rejected lots, called Average Outgoing Quality (AOQ); and

(ii) The average amount of inspection required for the rectifying inspection plan, called Average Total Inspection (ATI).

Suppose that incoming lots to the inspection activity have fraction defective p_0. Some of these lots will be accepted and others will be rejected. The rejected lots will be screened, and their final fraction defective will be zero. However, accepted lots have fraction defective p_0.Consequently, the outgoing lots from the inspection activity are a mixture of lots with fraction defective p_0 and fraction defective zero, so the average fraction defective in the stream of outgoing lots is p_1,(say) which is less than p_0. Thus, rectifying inspection program serves to correct the lot quality. This is demonstrated in the figure below.



Rectifying inspection programs are used in situations where the manufacturer wishes to know the average level of quality that is likely to result in at a given stage of the manufacturing processes. Thus, rectifying inspection programs are used either at receiving inspection, in-process inspection of semi finished products, or at final inspection of finished goods. The objective of in-plant usage is to give assurance regarding the average quality of material used in the next stage of the manufacturing operations.

Rejected lots may be handled in a number of ways. The best approach is to return rejected lots to the producer and require it to perform the screening and rework activities. This has the psychological effect of making the vendor responsible for poor quality and may exert pressure and the vendor/producer to improve its production processes or to install better process controls.

Average Outgoing Quality AOQ) is widely used for evaluation of a rectifying sampling plan. The average outgoing quality is the quality in the lot that results from the application of rectifying inspection. It is the average value of lot quality that would be obtained over a long sequence of lots from a process with fraction defective p.

13.6 INDIAN STANDARD TABLES AND THEIR APPLICATIONS

Selection of sampling plans by the technique of Indian Standard Institution is based on two considerations, viz., the cost of inspection and the quality of inspection desired by the producer and consumer. As explained in previous lessons, protection provided by a sampling plan to the producer and consumer is completely determined by its operating characteristic curve which provides the probabilities of accepting or rejecting a lot with varying proportion of defectives. The power of a sampling plan is decided by the steepness of its OC function. While operating characteristic function gives a complete profile of the protection afforded by the sampling plans but it has also to be noted that O.C. function does not give it as a single value which may serve as a measure of protection. In view of this, choice of a sampling plan is usually made with reference to certain specified points on the O.C. curve. Such as AQL, i.e., the maximum percent defectives may be considered as a satisfactory process average and LTPD, i.e., the percentage of defectives that can be tolerated in a lot, etc. The AQL/LTPD or such other values may be chosen on the basis of previous data available and also by an agreement between the parties concerned.

A comprehensive set of sampling plans classified in terms of AQL is given in the Indian Standard of Sampling Inspection Procedures, Part-I, for sampling procedures for inspection by attributes (IS : 2500, Part 1-2000, ISO 2859—I : 1999) and Part-II, inspection by variables for percent defectives (IS : 2500, Part-II—1965) published by. Bureau of Indian Standards, Manak Bhavan, New Delhi.

FURTHER SUGGESTED READINGS

IS:2500 Sampling Inspection Plans

These sampling inspection plans have been prepared by the Bureau of Standards. New Delhi and are being widely used. Part I of this set of plans is for lot-by-lot inspection. The inspection is by attributes of discrete items. The plans are indexed by Acceptable Quality Level (AQL) in terms of per cent nonconforming. The purpose is to maintain the specified AQL while providing an upper limit for the risk to the consumer of accepting occasional poor lot. These plans are intended primarily for a continuing series of lots sufficient to allow switching rules to be applied which provide for (1) an automatic the switching rules to be applied which provide protection to the consumer should a deterioration occur by tightened inspection or discontinuance of inspection and (2) an incentive to reduce inspection costs should consistently good quality be achieved. These plans may also be used for lots in isolation but in this case the OC curves should be consulted to find a plan to yield the desired protection. Sample sizes are designated by code letters for particular lot size and the prescribed inspection levels. Three types of plans - single, double and multiple (upto the 7th stage) are available.

Part II of this set is provided for single plans for lot-by-lot inspection by variables. Three types of plans are given which are applicable for the following situations: (a) variability known; (b) variability not known and estimated by sample s.d.(s); (c) variability not known and estimated by sample range (R or $\overline{R}$). The tables are for both (i) one-sided inspection where either upper specification limit U or lower specification limit L is given and (ii) two sided inspection where both upper and lower specification limits, U and L, are given.

Variability known: On the basis of the AQL and sample size code letter chosen from

Table I, the values of the sample size (n) and a factor k" are obtained from Table 2 and the lot is declared acceptable if

$\bar{x} + k''\sigma \leq U$ if U given or $\bar{x} - k''\sigma \geq L$, if L given.

For two sided specification limits, lot is declared acceptable if

(a) $\dfrac{\sigma}{U-L} \leq$ the maximum value specified in a table given for the chosen AQL and sample size code letter

(b) $\bar{x} + k''\sigma \leq U$ and $\bar{x} - k''\sigma \geq L$.

Variability unknown (σ -method): Here on the basis of AQL sample size code letter chosen from Table 1, the values of the sample size (n) and a factor k' are obtained from Table 3. The lot is accepted if $\bar{x} + k's \leq U$ if U is given or $\bar{x} - k's \geq L$ if L is given, s being the sample s.d..

For the two-sided case the lot is accepted if

(a) $\dfrac{s}{U-L} \leq$ maximum value provided in Table 5 for chosen AQL and sample size code letter or

(b) $\bar{x} + k's \le U$ and $\bar{x} - k's \ge L$

Variability unknown (R-method): On the basis of the chosen AQL and sample size code letter obtained from Table I, the sample size n and a factor k are obtained from Table 4. The range R is used if sample size is <10 and average range $\bar{R}$ is used if sample size is ≥ 10. The lot is accepted if

$\bar{x} + kR$ or $\bar{x} + k\bar{R} \le U$ if U is given or $\bar{x} - kR$ or $\bar{x} - k\bar{R} \ge L$ L if L given.

For the two-sided case, the lot is accepted if

(a) $\dfrac{R}{U-L}$ or $\dfrac{\bar{R}}{U-L} \le$ maximum value provided in Table 6 for chosen AQL and sample size code letter or

(b) $\bar{x} + kR$ or $\bar{x} + k\bar{R} \le U$ and $\bar{x} - kR$ or $\bar{x} - k\bar{R} \ge L$.

Provision for normal, reduced and tightened inspection is there for the observed state for quality of inspection lots.

13.7 SUMMARY AND FURTHER SUGGESTED READING

In accepting sampling inspection, an inspection lot is a group of articles accepted or rejected on the basis of one or more samples. How the lot is formed can influence the effectiveness of the acceptance sampling plan. The important considerations in forming lots for inspection are that the lots should be homogeneous. When lots are non-homogeneous, the acceptance sampling plan may not function as effectively as it could. Larger lots are preferred over smaller lots. It is usually more economically efficient to inspect large lots than small lots. Lots should be conformable to the materials handling systems used in both vendor and consumer facilities.

Whenever rejected lots are returned to the producers, the resubmission of these lots for another sampling inspection creates certain problems.

Accepting sampling plans usually require corrective action when lots are rejected. This generally takes the form of 100% inspection or screening of rejected lots, with all discovered defective either removed for subsequent rework or return to the producer.

Such sampling programs are called rectifying inspection programs, because the inspection activity affects the final quality of the outgoing product. Rectifying inspection program serves to correct the lot quality.

FURTHER SUGGESTED READING

1. D.C. Montgomery Introduction to Statistical Quality Control.

2. S. Biswas. Statistics of Quality Control.

3. Indian Standard Tables. Bureau of Indian Standards, Manak Bhavan, New Delhi.

13.8 SELF ASSESSMENT QUESTIONS

1. What is sampling inspection? Distinguish between the rectifying and the non-rectifying types.

2. Rectifying inspection program serves to correct the lot quality. Justify this statement.

3. How Indian Standard of Sampling Inspection Procedures are helpful in maintaining the AQL.

UNIT - IV

COMPUTATIONAL TECHNIQUES\

Lesson-14

14.1 Objectives

14.2 An introduction to computational techniques

14.3 Difference operators(Forward, backward and shift operators)

14.4 Relations between difference operators

14.5 Forward and backward difference tables

14.6 The problem of interpolation

14.7 Methods of interpolation

14.8 Summary and further suggested reading

Lesson - 15

15.1 Objectives

15.2 Newton's Forward and backward interpolation formula

15.3 Illustration of Newton's Forward and backward interpolation formula

15.4 Divided differences

15.5 Relation between forward difference and divided difference operators

15.6 Newton's divided difference formula

15.7 Lagrange's interpolation formula for divided differences

15.8 Inverse interpolation

15.9 Summary and further suggested reading

15.10 Self assessment questions

Lesson - 16

16.1 Objectives

16.2 Introduction to numerical integration and differentiation

16.3 General Quadrature formula

16.4 Trapezoidal Rule

16.5 Simpson's one third Rule

16.6 Simpson's Three Eighth Rule

16.7 Wddle's Rule

16.8 Summary and further suggested reading

16.9 Self assessment questions

Lesson - 17

17.1 Objectives

17.2 An introduction to numerical solution of equations

17.3 Method of false position

17.4 Newton-Rapson method

17.5 Method of iteration

17.6 Convergence of the Iteration and Newton-Rapson method

17.6 Summary and further suggested reading

14.1 OBJECTIVES

After studying this lesson you should be able to:

- Understand meaning of interpolation
- Know about finite differences and difference tables.
- Understand Newton's methods of interpolation.

14.2 AN INTRODUCTION TO COMPUTATIONAL TECHNIQUES

In many situations we are confronted with the mathematical and scientific problems which cannot be solved by the existing analytical methods or the solutions if they exist are so complex that they do not lead to any desired numerical information to conclude .In such cases the desired results may be obtained by pure numerical methods.

Thus numerical methods are concerned with the practical object of obtaining an approximate solution to the problem under consideration. This is correct to certain degree of accuracy. In the forthcoming lessons of this chapter we are going to discuss these computational techniques which can be effectively used for interpolation, extrapolation, numerical integration, differentiation and solution to linear equations etc.

14.3 DIFFERENCE OPERATORS (FORWARD, BACKWARD AND SHIFT OPERATORS)

FINITE DIFFERENCE TABLE: Let us consider a function

$y = f(x)$

Where x is an independent variable and y is a dependent variable. Suppose we are given equidistant values (finite in number) a, a+ h, a + 2h, a + 3h... of the variable x at an interval of 'h'. Then the corresponding values of the variable

$y = f(x)$ are $f(a), f(a+h), f(a+2h), f(a+3h)$……..

The values of the independent variable x are known as arguments and the corresponding values of the dependent variable y are called entries, e.g., f(a+h) is the entry corresponding to the argument a + h, and so on.

Forward Difference Operator Δ: This operator is of fundamental importance in the calculus of finite differences. By def.,

$$\Delta f(x) = f(x+h) - f(x) \quad ; \quad x = a, a+h, a+2h, a+3h \quad \ldots\ldots\ldots\ldots\ldots(1)$$

The equal interval of the arguments, viz., 'h' is known as the interval of differencing. In particular,

$$\Delta f(a) = f(a+h) - f(a), \ \Delta f(a+h) = f(a+2h) - f(a+h) \text{ and so on}$$

The differences defined in (1) are known as first order differences. If the operation of Δ is again performed on differences obtained in (1) we get the second order differences, which are denoted by Δ^2. Thus

$$\Delta^2 f(x) = \Delta [f(x+h) - f(x)]$$

$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

Where x= a, a+ h, a + 2h, a + 3h……

In particular,

$$\Delta^2 f(a) = \Delta [f(a+h) - f(a)]$$

$$= [f(a+2h) - f(a+h)] - [f(a+h) - f(a)]$$

$$= \Delta f(a+h) - \Delta f(a)$$

and so on. The third order difference denoted by Δ^3 are obtained by taking the differences of the second order differences obtained in above equation Proceeding similarly e get higher order differences.

Backwrd Differences: Suppose we are given equidistant values (finite in number) a, a+ h, a + 2h, a + 3h... of the variable x at an interval of 'h'.

The backward difference operators are denoted by ∇ **and is given by**

∇ f(x+h)= [f(x+h)- f(x)] = Δ f(x) ; x = a, a+ h, a + 2h, a + 3h……

Thus the backward difference of f(x+h) is the same as the forward difference of f(x)

The Shift Operator E. The operator E is defined as

$$E.f(x) = f(x+h) - f(x)$$

i.e., it results in increasing the argument by the interval of differencing.

Thus

$$E^2 f(x) = E\left[E\{f(x)\}\right] = E[f(x+h)] = f(x+2h)$$

Similarly

$$E^3 f(x) = E^2\left[E\{f(x)\}\right] = E^2[f(x+h)] = f(x+3h)$$

In general

$$E^r f(x) = f(x+rh)$$

14.4 RELATIONS BETWEEN DIFFERENCE OPERATORS

Relation between E and ∇. By definition

$$\nabla f(x+h) = \left[f(x+h) - f(x)\right]$$

$$\Rightarrow f(x) = \left[f(x+h) - \nabla f(x)\right] \text{ f(x)]}$$

$$= (1+\nabla) f(x+h)$$

Also we have

$$f(x) = E^{-1} f(x+h)$$

or $E^{-1} f(x+h) = (1+\nabla) f(x+h)$

or $\mathbf{1+\nabla = E^{-1}}$

Relation between E and Δ: By definition

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x) \\ &= Ef(x) - f(x) = [E-1] f(x)\end{aligned}$$

or $\mathbf{\Delta = [E-1]}$

14.5 FORWARD AND BACKWARD DIFFERENCE TABLES

Let us consider a function

Where x is an independent variable and y is a dependent variable. Suppose we are given equidistant values (finite in number) a, a+ h, a + 2h, a + 3h... of the variable x at an interval of 'h'. Then the corresponding values of the variable

y = f(x) are f(a), f(a+h),f(a+2h),f(a +3h)……..

The values of the independent variable x are known as arguments and the corresponding values of the dependent variable y are called entries, e.g., f(a+h) is the entry corresponding to the argument a + h, and so on. Then the forward difference table is given by

The forward Difference table

argument	Entry	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
A	**f(a)**	f(a+h) - f(a) = **Δ f(a)**	Δ f(a+h) - Δ f(a) = $\mathbf{\Delta^2 f(a)}$	Δ^2 f(a+h) $-\Delta^2$f(a) = $\mathbf{\Delta^3 f(a)}$	Δ^3 f(a+h) $-\Delta^3$ f(a) = $\mathbf{\Delta^4 f(a)}$
a+h	f(a+h)	f(a+2h)- f(a+h) = Δ f(a+h)	Δ f(a+2h) - Δ f(a+h) = Δ^2 f(a+h)	Δ^2 f(a+2h) - Δ^2 f(a+h) = Δ^3 f(a+h)	
a+2h	f(a+2h)	f(a+3h) - f(a+2h) = Δ f(a+2h)	Δ f(a+3h) - Δ f(a+2h) = Δ^2 f(a+2h)		
a+3h	f(a +3h)	f(a+4h) - f(a+3h) = Δ f(a+3h)			
a+4h	f(a +4h)				

Here f(a) is known as first entry and h is called interval of differencing

Example : We draw up the forward difference table for the data given below

X	1	2	3	4	5	6
f(x)	17	2	1	2	17	75

X	f(x)	Δ f(x)	Ä^{2} f(x)	Δ^3 f(x)	Δ^4 f(x)
1	17				
		-15			
2	2		14		
		-1		-12	
3	1		2		24
		1		12	
4	2		14		19
		15		31	
5	17		45		
		60			
6	77				

The backward difference table is given by

Backward Difference Table

argument	Entry	∇ f(x)	$\nabla^2 f(x)$	∇^3 f(x)	∇^4 f(x)	∇^5 f(x)
A	**f(a)**					
a+h	f(a+h)	f(a+h)- f(a) = ∇ **f(a+h)**				
a+2h	f(a+2h)	f(a+2h)- f(a+h) = ∇ f(a+2h)	∇ f(a+2h)- ∇ f(a+h) = ∇^2 f(a+2h)			
a+3h	f(a+3h)	f(a+3h)- f(a+2h) = ∇ f(a+3h)	∇ f(a+3h)- ∇ f(a+2h) = ∇^2 f(a+3h)	∇^2 f(a+3h) - ∇ f(a+2h) = ∇^3f(a+3h)		
a+4h	f(a+4h)	f(a+4h)- f(a+3h) = ∇ f(a+4h)	∇ f(a+4h)- ∇ f(a+3h) = ∇^2 f(a+4h)	∇^2 f(a+4h) - ∇^2 (a+3h) = ∇^3f(a+4h)	∇^3f(a+4h) - ∇^3f(a+3h) = ∇^4 f(x+4h)	
a+5h	f(a+5h)	f(a+5h)- f(a+4h) = ∇ f(a+5h)	∇ f(a+5h)- ∇ f(a+4h) = ∇^2 f(a+5h)	∇^2 (a+5h) - ∇^2 (a+4h) = ∇^3f(a+5h)	∇^3 f(a+5h) - ∇^3f(a+4h) = ∇^4 f(x+5h)	∇^4 f(x+5h) - ∇^4 f(x+4h) = ∇^5 f(x+5h)

14.1 THE PROBLEM OF INTERPOLATION

'**Interpolation'** this term usually denotes the process of finding the intermediate value of a function from a set of given values of that function. For example, using the following given values of x and y

X	4	6	8	10	12
Y	13	29	82	119	198

We may be required to find, i.e. interpolate, the value of y when x = 4.

Thiele- defines interpolation as "**the art of reading between the lines of a table."**

The interpolation may be defined as the ***"technique of obtaining the most likely estimate of a certain quantity under certain assumptions."***

"Interpolation is the estimation of a most likely estimate in given conditions. The technique of estimating a past figure is termed as interpolation, while that of estimating a probable figure fore the future is called extrapolation." — Hirsch

Assumptions of Interpolation. The following are the fundamental assumptions in any method of interpolation:

(i) There are no sudden jumps or falls in the values of the entries for the period under consideration. In other words, the given data does not refer to abnormal periods such as period of famines, wars, epidemics, etc.

Mathematically, it means that the data can be represented by a smooth or continuous curve which implies that given data can be represented by a polynomial of certain degree, which is determined by the following theorem.

One and only one polynomial curve of degree less than or equal to n passes through a given set of (n + 1) distinct points.

(ii) In the absence of the evidence to the contrary, the rise or fall in the data has been uniform.

Uses of Interpolation.

1. The need for interpolating missing observations or making forecasts or projections

arises in a number of disciplines like economics, business, social sciences, actuarial work, population studies, etc

2. Interpolation techniques are used to fill in the gaps in the statistical data for the sake of continuity of information. These gaps in the data may be due to the following reasons:

   (i) Due to certain financial and organizational difficulties, data may not be collected on census basis and sampling techniques may be used to obtain the relevant information. The intermediate gaps are then filled by interpolation methods.

   (ii) Data for some periods may not be collected due to certain unavoidable circumstances.

   (iii) Figures of some of the periods may be erased, destroyed or lost due to certain reasons.

Interpolation techniques help us to obtain the best (most likely) substitutes for the original missing values under certain assumptions

14.2 METHODS OF INTERPOLATION

Methods of interpolation: Let us consider a function

$y = f(x)$

where x is an independent variable and y is a dependent variable. Suppose we are given equidistant values (finite in number) a, a+ h, a + 2h, a + 3h... of the variable x at an interval of 'h'. Then the corresponding values of the variable $y = f(x)$ are f(a), f(a+h),f(a+2h),f(a +3h)……..

The values of the independent variable x are known as arguments and the corresponding values of the dependent variable y are called entries, e.g., f(a+h) is the entry corresponding to the argument a + h, and so on.

There are various methods of finding f(x) or interpolate the values of y for given values of x. In graphical method we smooth out to form a curve which this polynomial should represented and the values are interpolated.

When the values of x are **not equally spaced** we can use method of curve fitting Lagrange's method, Newton's divided differences formulae can be used.

In case values of arguments are equally spaced ,the other methods which can be applied are

(i) Binomial expansion method for interpolation the missing values.

(ii) Newton's advancing differences formula where the values to be interpolated is nearer to the beginning.

(iii) Newton's backward differences formula where the values to be interpolated is nearer to the last value and so on

Fundamental Theorem of Finite Differences: If f(x) is a polynomial of nth degree in x, then

$$\begin{aligned}\Delta^r f(x) &= constant, \quad && r = n \\ &= 0, && r > n\end{aligned}$$

In other words, the nth order difference of a polynomial of nth degree is constant and higher differences are zero.

Proof. Let f(x) be a polynomial of nth degree given by

$$f(x) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \ldots\ldots + a_{n-1} x + a_n \qquad (a \neq 0)$$

where a_o, a_1, ..., a_2, are constants.

By def.,

$$\Delta f(x) = f(x+h) - f(x)$$

$$= \left[a_0 (x+h)^n + a_1 (x+h)^{n-1} + a_2 (x+h)^{n-2} + \ldots\ldots. + a_{n-1}(x+h) + a_n\right]$$

$$- \left[a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n\right]$$

$$= \Big[a_0 (x^n + \overset{n}{\underset{1}{c}} x^{n-1} h + \overset{n}{\underset{2}{c}} x^{n-2} h^2 + \ldots. + h^n) + a_1 (x^{n-1} + \overset{n-1}{\underset{1}{c}} x^{n-2} h + \overset{n-1}{\underset{2}{c}} x^{n-3} h^2 + \ldots. + h^{n-1})$$

where b_2, b_3, ..., bn are constants independent of x.

Thus the first order difference of a polynomial of degree n is a polynomial of degree (n - 1).

Again

$$\Delta^2 f(x) = \left[a_o nh(x+h)^{n-1} + b_2(x+h)^{n-2} + b_3(x+h)^{n-3} + \ldots\ldots + b_{n-1}(x+h) + b_n\right]$$

$$-\left[a_o nhx^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \ldots\ldots + b_{n-1}x + b_n\right]$$

$$= \left[a_0 nh\left\{\overset{n-1}{\underset{1}{c}}\, x^{n-2}h + \overset{n-1}{\underset{2}{c}}\, x^{n-3}h^2 + \ldots. + h^{n-1}\right\} + b_2\left\{x^{n-2} + \overset{n-2}{\underset{1}{c}}\, x^{n-3}h + \overset{n-2}{\underset{2}{c}}\, x^{n-4}h^2\right.\right.$$

$$\left.\left. + \ldots. + h^{n-2}\right\} + \ldots\ldots + b_{n-1}x + b_n\right]$$

where are constants independent of x.

Thus $\Delta 2$ f(x)= A polynomial in x of (n-2)th degree.

Similarly proceeding, we shall get

Hence the nth order difference of a polynomial of nth degree is constant and higher order differences are all zero.

Then

$\Delta^{n+1} f(x) = \Delta(k) = 0 \; and \; \Delta^{n+2} f(x) = \ldots\ldots = 0$ Hence the theorem

Example: Evaluate

$(a) \qquad \Delta^n (ax^n + bx^{n-1})$

(b) Δ10 (1-ax) (1 -bx2) (1-cx3) (1-dx4)

Solution

(a)

$$\Delta^n(ax^n + bx^{n-1}) = \Delta^n(ax^n) + \Delta^n(bx^{n-1})$$
$$= a\Delta^n(x^n) + b\Delta^n(x^{n-1})$$
$$= an! + b.0 = an!$$

{By using the fundamental theorem of finite differences}

(b) Δ10 (1-ax) (1-bx2) (1-cx3) (1-dx4) $= \Delta^{10}(abcd.x^{10}) = bcd\Delta^{10}(.x^{10})$

$$= abcd.10!$$

{Since $\Delta^n f(x) = a_0 n(n-1)(n-2).......2.1h^n x^{n-n} = a_o(n!)h^n$ }

Missing terms (Equal intervals): Sometimes we may be given a set of equidistant terms with some terms (one or two or more) missing. The problem of estimating such terms can be easily tackled by the use of the operators E and Δ.

Let us suppose that we are given (n+1) equidistant arguments, (x = 0, 1, 2, ..., n, say) but the entry f(r) corresponding to any one of them say $(r + l)^{th}$ argument is not given and we want to estimate that. Since we are given n entries, the data can be represented by a polynomial of $(n\text{-}l)^{th}$ degree. Hence we may take f(x) as a polynomial of degree (n-1).

So that we can take

$\Delta^{n-1} f(x) =$ Constant and $\Delta^n f(x) = 0$; x = 0, 1, 2. .... n, ...(*)

In particular

$$\Delta^n f(0) = 0,$$

i.e., $[E-1]^n f(0) = 0$ {as Δ=[E-1]}

$$\left[E^n - \overset{n}{\underset{1}{C}} E^{n-1} + \overset{n}{\underset{2}{C}} E^{n-2} + \overset{n}{\underset{3}{C}} E^{n-3} + ... + (-1)^n\right] f(0) = 0$$

$$\Rightarrow \qquad f(n) - \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right] f_{(n-1)} + \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right] f_{(n-2)} + ... + (-1)^n f_{(0)} = 0$$

From this equation, the missing entry can be easily calculated.

If, in a set of (n + 2) equidistant arguments, two entries are missing, then we have from (*), $\Delta^n f(0) = 0$ and $\Delta^n f(1) = 0$

i.e., $[E-1]^n f(0) = 0$

and $[E-1]^n f(1) = 0$

Expanding and simplifying as above, the two missing terms can be estimated by solving the above equations.

Example: Estimate $f_{(2)}$ from the following table:

X:	1	2	3	4	5
f(x).	7	?	13	21	37

Solution. Since we are given four entries, viz., $f_1, f_3\ f_4$ *and* f_5, the function f(x) can be represented by a third degree polynomial.

$\Delta^3 f(x) = constant$ and $\Delta^4 f(x) = 0$

In particular

$$\lfloor E^4 - 4E^3 + 6E^2 - 4E + 1 \rfloor f(1) = 0$$

$$\Rightarrow \quad f_5 - 4f_4 + 6f_3 - 4f_2 + f_1 = 0$$

$$\Rightarrow 37 - 4 \times 21 + 6 \times 13 - 4 f(2) + 7 = 0$$

or $f_2 = 38/4 = 9.5$

Method of Parabolic Curve Fitting: The form of function y =f(x) or its estimate for any given value of x can be obtained by fitting a polynomial curve to the given set of observations provided the values of x (arguments) are at equal intervals.

The method is based on the fundamental theorem of algebra viz.," One and only one polynomial curve of degree less than or equal n passes through a given set of (n + 1) distinct points."

Thus if we are given (n + 1) equidistant arguments and entries then we can represent the function y =f (x) by a polynomial of nth degree,

$$y = f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + ...a_{n-1}x + a_n$$
$$\ldots\ldots\ldots\ldots\ldots(*)$$

where a_0, a_1 ..., a_n are (n + 1) constants whose values are to be determined from the (n+1) equations obtained on substituting the given values of x and y =f(x) in (*).

Solving the (n + 1) equations so obtained and substituting the values of a_0, a_1, ..., a_n in (*) we get the required form of function y =f (x), which can then be used to estimate y for any given value of x.

Example: Find f(x), given that f(0)= - 3,f(1) = 6,f(2) = 8,f(3) = 12. State your assumption, if any. Hence find f(6).

Sol: Since we are given 4 entries, we can approximate f(x) by a polynomial of 3rd degree, say, $f(x) = ax^3 + bx^2 + cx + d$ . ... (*)

Putting x= 0, 1, 2, and 3 in (*) we get respectively:

$$f(0) = d = -3,$$

$$f(1) = a + b + c + d = 6$$

$$f(2) = 8a + 4b + 2c + d = 8,$$

$$f(3) = 27a + 9b + 3c + d = 12$$

Solving for these equations we get $a = \frac{3}{2}$, $b = 8$, $c = \frac{21}{2}$, $d = 3$

Finally, substituting the values of a, b, c and (d) in (*),we get the form of function f (x) as:

$$f(x) = \frac{3}{2}x^3 - 8x^2 + \frac{31}{2}x - 3 \qquad \ldots\ldots\ldots\ldots (**)$$

Putting x =6 in (**) we get

$$f(6) = \frac{3}{2}6^3 - 8(6)^2 + \frac{31}{2} \times 6 - 3 = 126$$

14.1 SUMMARY AND FURTHER SUGGESTED READING

The main purpose of studying this lesson was to understand meaning of interpolation and to have knowledge about the finite differences and difference tables. To understand Newton's methods of interpolation. The major outcomes of this lesson are as under

- Interpolation: Interpolation is the estimation of a most likely estimate in given conditions. (in a certain range). The technique of estimating a past figure is termed as interpolation.
- Extrapolation: The technique of estimating a probable figure for the future is called extrapolation.
- Forward difference operator: Δ f(x) = f(x+h)- f(x) ; x= a, a+ h, a + 2h, a + 3h
- Back ward difference operator and shift operator:

  Back ward difference operator : ∇ f(x+h)= [f(x+h)- f(x)] = Δ f(x)

  Shift operator: The operator E is defined as $E.f(x) = f(x+h) - f(x)$

FURTHER SUGGESTED READING

1. Fundamentals of Mathematical Statistics S. Chand and Co., New Delhi.
2. Scarborough, J.B. Numerical Mathematical Analysis, Oxford University Press.

15.1 OBJECTIVES

Main objectives of studying this lesson are:

- To understand Newton's Forward and backward interpolation formulae
- To know the basic concept of divided differences
- To have an idea about relation between forward and divided differences
- To have conceptual understating of Newton's methods of interpolation for divided differences
- To have idea about Lagrange's methods of interpolation for divided differences

15.2 NEWTON'S FORWARD AND BACKWARD INTERPOLATION FORMULAE

Gregory Newton's (Forward Interpolation) Formula for Equal Intervals,

$$f_{(a+xh)} = f_{(a)} + \binom{x}{1}\Delta f_{(a)} + \binom{x}{2}\Delta^2 f_{(a)} + \ldots\ldots + \binom{x}{r}\Delta^r f_{(a)} + \ldots. \quad (1)$$

where 'a' is the first argument and h is the common interval of differencing, last term depending on the degree of the polynomial f(x).

Proof. By def.

$$f(a+xh) = E^X f(a) = (1+\Delta)^X f(a) \qquad \{\text{as } \Delta=[E-1]\}$$

$$= [1 + \binom{x}{1}\Delta + \binom{x}{2}\Delta^2 + \ldots\ldots + \binom{x}{r}\Delta^r + \ldots.. + \Delta^x]\, f_{(a)}$$

$$= f_{(a)} + \binom{x}{1}\Delta f_{(a)} + \binom{x}{2}\Delta^2 f_{(a)} + \ldots\ldots + \binom{x}{r}\Delta^r f_{(a)} + \ldots. + \Delta^x f_{(a)}$$

which is the required formula.

Newton's Forward Difference formula is useful when the value to be interpolated lies towards the beginning of the given data.

Gregory-Newton's Backward Interpolation Formula: This formula is based on the backward differences ∇ .According to this formula

$$f_{b+xh} = f_b + \binom{x}{1}\nabla f_b + \frac{x(x+1)}{2!}\nabla^2 f_b + \frac{x(x+1)(x+2)}{3!}\nabla^3 f_b + \\ \dots + \frac{x(x+1)(x+2).....(x+n-1)}{n!}\nabla^n f_b$$

Where b is the last argument in the difference table

Proof: If b = a+nh, is the last argument in the difference table, then by def., we have

$$f_{b+xh} = E^x f_b = (1-\nabla)^{-x} f_b \qquad \because E = (1-\nabla)^{-1}$$

$$= f_b + \binom{x}{1}\nabla f_b + \frac{x(x+1)}{2!}\nabla^2 f_b + \frac{x(x+1)(x+2)}{3!}\nabla^3 f_b + \dots\dots \qquad (1)$$

the last term depending on the degree of the polynomial f_{a+xh} .

If f_{a+xh} is a polynomial of nth degree, $\Delta^r f_a = 0$, r> n and hence the series (1) terminates after (n+1) terms, thus giving

$$f_{b+xh} = f_b + \binom{x}{1}\nabla f_b + \frac{x(x+1)}{2!}\nabla^2 f_b + \frac{x(x+1)(x+2)}{3!}\nabla^3 f_b + \\ \dots + \frac{x(x+1)(x+2).....(x+n-1)}{n!}\nabla^n f_b$$

which is the required Newton's Backward Difference formula. where x is given by

$$x = \frac{Period\ \ of\ \ interpolation - Last\ \ argument}{Interval\ \ of\ \ differencing}$$

Newton's Backward Difference formula is useful when the value to be interpolated lies towards the end of the given data.

15.3 ILLUSTRATION OF NEWTON'S FORWARD AND BACKWARD INTERPOLATION FORMULA

Illustration: Estimate the missing figure from the following table

X:	20	22	25	30	35
Y:	73	?	198	573	1198

Solution: As we are given four values the given function approximates to polynomial of degree three and as the values of x are equidistant Newton's forward difference formula can be applied.

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	73			
		125		
25	198		250	
		375		0
30	573		250	
		625		
35	1198			

Here we want

$$f(22) = f(x+h) \qquad (say)$$

$$\therefore \qquad a + xh = 22 \qquad \Rightarrow \qquad 20 + x \times 5 = 22$$

$$or \qquad x = \frac{2}{5} = 0.4$$

Substituting x=0.4 in Newton's formula given by

$$f_{(a+xh)} = f_{(a)} + \binom{x}{1}\Delta f_{(a)} + \binom{x}{2}\Delta^2 f_{(a)} + \dots\dots + \binom{x}{r}\Delta^r f_{(a)} + \dots.$$

We get

$$y_{22} = y_{20} + x\Delta y_{20} + x\left(\frac{x(x-1)}{2!}\right)\Delta^2 y_{20} + x\left(\frac{x(x-1)(x-2)}{3!}\right)\Delta^3 y_{20}$$

$$= 73.0 + 0.4 \times 125 + \frac{.0.4(0.4-1)}{2\times 1} \times 250 + 0$$

$$= 73.0 + 50.0 - 30.0 = 93$$

Example : From the following table of yearly installments of premiums for policies maturing at quinquenial ages, estimate the premiums for policies maturing at the ages 46 years.

Age(x):	45	50	55	60	65
Premium f(x):	2.8710	2.4040	2.0838	1.8620	1.7120

Sol: Table of Differences

Age x	Premium f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
45	2.8710	-0.467			
50	2.4040	-0.321	0.146	-0.046	
55	2.0838	-0.221	0.100	-0.029	0.017
60	1.8620	-0.150	0.071		
65	1.7120				

We want

$$f(46) = f(x+h) \qquad (say)$$

$\therefore \quad a + xh = 46 \quad \Rightarrow \quad 45 + x \times 5 = 46$

$or \quad x = \frac{1}{5} = 0.5$

Substituting x=1/5 in Newton's formula given by

$$f_{(a+xh)} = f_{(a)} + \binom{x}{1}\Delta f_{(a)} + \binom{x}{2}\Delta^2 f_{(a)} + \ldots\ldots + \binom{x}{r}\Delta^r f_{(a)} + \ldots.$$

$$f(46) = f(45) + \frac{1}{5}\Delta f(45) + \frac{1}{5}\left(\frac{-4}{5}\right)\frac{1}{2!}\Delta^2 f(45) + \frac{1}{5}\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\frac{1}{3!}\Delta^3 f(45)$$

$$+\frac{1}{5}\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\left(-\frac{14}{5}\right)\frac{1}{4!}\Delta^4 f(45)$$

$$= 2.871 + \frac{1}{5}(-0.467) - \frac{2}{25}(0.146) + \frac{6}{125}(-0.046) - \frac{21}{625}(0.017)$$

$$= 2.871 - 0.934 - 0.02336 - 0.0013248 - 0.000506 = 2.753(approx.)$$

Illustration The following table gives the census population of a town for the years 1961 to 2001. Estimate the population for the year 1995 by using on appropriate interpolation formula.

Year :	1961	1971	1981	1991	2001
Population (in lakhs):	36	66	81	93	101

Solution. Since the value to be interpolated lies towards the end of the given data, we shall use Newton's backward difference formula.

Backward Difference Table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1961	46				
		20			
1971	66		-5		
		15		2	
1981	81		-3		-3
		12		-1	
1991	93		-4		
		8			
2001	101				

In the usual notations of we have

$$x = \frac{Period\ \ of\ interpolation - Last\ argument}{Interval\ \ of\ \ differencing}$$

$$x = \frac{1995 - 2001}{10} = -0.6$$

The leading backward differences of last entry are given respectively by 8, -4, -1 and -3. Substituting in Newton's backward difference formula), we get

$$f_{b+xh} = f_b + \binom{x}{1}\nabla f_b + \frac{x(x+1)}{2!}\nabla^2 f_b + \frac{x(x+1)(x+2)}{3!}\nabla^3 f_b + \\ \quad .... + \frac{x(x+1)(x+2).....(x+n-1)}{n!}\nabla^n f_b$$

$$f_{1995} = 101 - 0.6 \times 8 + \frac{-0.6(-0.6+1)}{2!} \times (-4) + \frac{-0.6(-0.6+1)(-0.6+2)}{3!} \times (-1) \\ + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times (-3)$$

$$= 101 - 4.8 - 0.48 + 0.056 + 0.1008 = 96.937\ lakhs$$

15.3 DIVIDED DIFFERENCES

INTERPOLATION WITH ARGUMENTS AT UNEQUAL INTERVALS

The operates E, Δ, ∇ and Newton's-Gregory formulae of interpolation can be used only if we are given the entries corresponding to equidistant values of the argument and in the case of unequal intervals , the operators E and A don't serve our purpose and we define a more general class of differences, known as divided differences.

Divided Differences: Let $f_{a0}, f_{a1}, f_{a2}, ..., f_{an}$, be the entries corresponding to the arguments a_0, a_1, a_2 ..., a_n, which need not necessarily be equidistant. Then the first order divided difference of f(x), for the arguments a_0, a_1 denoted by $f(a_0, a_1)$ and is given by

$$f(a,b)=\frac{f(b)-f(a)}{b-a}=\frac{f(a+h)-f(a)}{h}=\frac{1}{h}\Delta f(a) \qquad \text{..................(*)}$$

Here we see that $\quad f(a_1,a_0)=\dfrac{f(a_0)-f(a_1)}{a_0-a_1}=\dfrac{f(a_1)-f(a_0)}{a_1-a_0}$

$$\Rightarrow \qquad f(a_0,a_1)=f(a_1,a_0) \qquad \text{..................(2)}$$

The second order divided difference of f(x) for the arguments a_0, a_1 and a_2 is defined as

$$f(a_0,a_1,a_2)=\frac{f(a_2,a_1)-f(a_1,a_0)}{a_2-a_0}$$

$$=\frac{1}{a_2-a_0}\left(\frac{f(a_2)-f(a_1)}{a_2-a_1}-\frac{f(a_1)-f(a_0)}{a_1-a_0}\right)$$

$$=\frac{1}{a_2-a_0}\left(\frac{f(a_1)}{a_2-a_1}+\frac{f(a_0)}{a_1-a_0}-\frac{(a_2-a_0)f(a_1)}{(a_2-a_1)(a_1-a_0)}\right)$$

$$=\left(\frac{f(a_0)}{(a_0-a_1)(a_0-a_2)}+\frac{f(a_1)}{(a_1-a_0)(a_1-a_2)}+\frac{f(a_2)}{(a_2-a_1)(a_2-a_0)}\right) \qquad \text{......(3)}$$

and so on.

Thus, the order of a divided difference is one less than the number of arguments required for its definition.

15.4 RELATION BETWEEN FORWARD DIFFERENCE AND DIVIDED DIFFERENCE OPERATORS

Relation between forward difference operator and divided difference operator

Let us take the arguments $a,b,c,d,$ etc to be at equal interval of h, (say), so that

$b=a+h, c=a+2h, d=a+3h,$ and so on.

Then by definition,

$$f(a,b)=\frac{f(b)-f(a)}{b-a}=\frac{f(a+h)-f(a)}{h}=\frac{1}{h}\Delta f(a) \qquad \text{.................(*)}$$

Similarly

$$f(b,c)=\frac{f(c)-f(b)}{c-b}=\frac{f(a+2h)-f(a)}{h}=\frac{1}{h}\Delta f(a+h) \qquad \text{.................(**)}$$

$$f(a,b,c)=\frac{f(b,a)-f(c,a)}{c-a}=\frac{1}{2h}\left[\frac{1}{h}\left\{\Delta f(a+h)-\Delta f(a)\right\}\right]$$

$$=\frac{1}{2h^2}\left[\Delta\left\{f(a+h)-f(a)\right\}\right]=\frac{1}{2h^2}\Delta^2 f(a) \qquad From\ \ (*)\ and\ \ (**)$$

Where the second order divided difference of $f(x)$, for the arguments a,b,c denoted by $f(a,b,c)$.Similarly

$$f(b,c,d)=\frac{1}{2h^2}\Delta^2 f(a+h)$$

$$f(a,b,c,d)=\frac{f(b,c,d)-f(a,b,c)}{d-a}=\frac{1}{3h}\left[\frac{1}{2h^2}\left(\Delta^2 f(a+h)-\Delta^2 f(a)\right)\right]$$
$$=\frac{1}{3!h^3}\Delta^2\left(f(a+h)-f(a)\right)=\frac{1}{3!h^3}\Delta^3 f(a)$$

Where the third order divided difference of $f(x)$, for the arguments a,b,c,d denoted by $f(a,b,c,d)$

Proceeding similarly the rth order divided difference in the terms of forward difference is given by

$$The\ rth\ order\ divided\ difference = f(a,b,c,d,.....,(r+1)terms)=\frac{1}{r!h^r}\Delta^r f(a)$$

15.3 NEWTON'S DIVIDED DIFFERENCE FORMULA

Newton's Divided Difference Formula.

By definition, we have

$$f(x,a)=\frac{f(x)-f(a)}{x-a} \qquad \Rightarrow f(x)=f(a)+(x-a)f(x,a) \qquad \ldots\ldots\ldots\ldots\ldots(*)$$

Where the first order divided difference of, for the arguments denoted by The second order divided difference for arguments is given by

Substituting in (*), we get

$$\begin{aligned} f(x) &= f(a)+(x-a)\left[f(a,b)+(x-b)f(x,a,b)\right] \\ &= f(a)+(x-a)f(a,b)+(x-a)(x-b)f(x,a,b) \qquad \ldots\ldots\ldots(**) \end{aligned}$$

Where the second order divided difference of $f(x)$, for the arguments x,a,b denoted by $f(x,a,b)$

The third order divided difference for the arguments x,a,b,c is given by

$$f(x,a,b,c)=\frac{f(x,a,b)-f(a,b,c)}{x-c}$$

$$\therefore \qquad f(x,a,b)=f(a,b,c)+(x-c)f(x,a,b,c)$$

Substituting in (**), we get

$$f(x)=f(a)+(x-a)f(a,b)+(x-a)(x-b)\left[f(a,b,c)+(x-c)f(x,a,b,c)\right]$$

$$=f(a)+(x-a)f(a,b)+(x-a)(x-b)f(a,b,c)+(x-a)(x-b)(x-c)f(x,a,b,c)$$

Where $f(x,a,b,c)$ denote the third order divided difference for the arguments x,a,b,c

Similarly proceeding, we shall get

$$f(x)=f(a)+(x-a)f(a,b)+(x-a)(x-b)f(a,b,c)+(x-a)(x-b)(x-c)f(x,a,b,c)$$

$$+(x-a)(x-b)(x-c)(x-d)f(x,a,b,c,d)+\ldots\ldots \qquad (***)$$

If $f(x)$ is a polynomial of nth degree then rth order divided difference is zero for , $r > n$ and the series (***) terminates after (n + 1) terms ,we get the required Newton's divided difference interpolation formula.

Example: The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable?

x	f(x)	Ist order divided difference	2nd order divided difference	3rd order divided difference
3	168			
		$\frac{120-168}{7-3}=-12$		
7	120		$\frac{-24+12}{9-3}=-2$	
		$\frac{72-120}{9-7}=-24$		$\frac{5+2}{10-3}=1$
9	72		$\frac{-9+24}{10-7}=5$	
		$\frac{63-72}{10-9}=-9$		
10	63			

Hence, using Newton's divided difference interpolation formula, we get

$$f(x) = f(a) + (x-a)f(a,b) + (x-a)(x-b)f(a,b,c) + (x-a)(x-b)(x-c)f(x,a,b,c)$$

$$= 168 - 12(x-3) - 2(x-3)(x-7) + 1(x-3)(x-7)(x-9)$$

$$= x^3 - 21x^2 + 119x - 27$$

When x = 6, the estimate of the function is given by:

$$f6) = 6^3 - 21\times 6^2 + 119\times 6 - 27 = 147$$

15.3 LAGRANGE'S INTERPOLATION FORMULA FOR DIVIDED DIFFERENCES

Lagrange's Interpolation Formula: Let fa_0, fa_1, fa_2 ,..., fa_n be the (n + 1) entries corresponding to the arguments $a_0, a_1, a_2 \ldots, a_n$, which are not necessarily equally spaced. Then the function f(x) can be approximated by a polynomial of the nth degree.

Let us consider the divided difference $f(x, a_0, a_1, \ldots\ldots, a_n)$ corresponding to (n+1) arguments $x, a_0, a_1, \ldots\ldots, a_n$ .since the order of divided difference is one less than the number of arguments required for its definition, is a divided difference of order (n+1). Further is a polynomial of degree n so that the rth order divided difference of
Hence

$$\Rightarrow \quad \frac{f(a_0)}{(a_0 - x)(a_0 - a_1)(a_0 - a_2)\ldots\ldots(a_0 - a_n)} + \frac{f(a_1)}{(a_1 - x)(a_1 - a_0)(a_1 - a_2)(a_1 - a_n)}$$

$$+ \frac{f(a_2)}{(a_2 - x)(a_2 - a_0)(a_2 - a_1)\ldots\ldots\ldots(a_2 - a_.)} + \ldots. + \frac{f(a_n)}{(a_n - x)(a_n - a_0)(a_n - a_1)\ldots\ldots\ldots(a_n - a_{n-1.})}$$

$$+ \frac{f(x)}{(x - a_0)(x - a_1)(x - a_2)\ldots\ldots\ldots(x - a_{n.})} = 0$$

$$\Rightarrow \quad \frac{f(x)}{(x - a_0)(x - a_1)(x - a_2)\ldots\ldots\ldots(x - a_{n.})} = \frac{f(a_0)}{(x - a_0)(a_0 - a_1)(a_0 - a_2)\ldots\ldots(a_0 - a_n)}$$

$$\frac{f(a_1)}{(a_1 - x)(a_1 - a_0)(a_1 - a_2)(a_1 - a_n)} + \ldots + \frac{f(a_n)}{(x - a_n)(a_n - a_0)(a_n - a_1)\ldots\ldots\ldots(a_n - a_{n-1.})}$$

$$\Rightarrow \quad f(x) = \frac{(x - a_1)(x - a_2)\ldots\ldots\ldots(x - a_{n.})}{(a_0 - a_1)(a_0 - a_2)\ldots\ldots(a_0 - a_n)} f(a_0) + \frac{(x - a_0)(x - a_2)\ldots\ldots\ldots(x - a_{n.})}{(a_1 - x)(a_1 - a_0)(a_1 - a_2)(a_1 - a_n)} f(a_1)$$

$$+ \ldots\ldots\ldots + \frac{(x - a_0)(x - a_1)(x - a_2)\ldots\ldots\ldots(x - a_{n-1.})}{(a_n - a_0)(a_n - a_1)\ldots\ldots\ldots(a_n - a_{n-1.})} f(a_n)$$

which is the Lagrange's formula of interpolation.
Illustration: By using Lagrange's interpolation formula x=656. (Retain for decimal places in your answer) for the below given data.

$x:$	654	658	659	661
$f(x):$	2.8156	2.8182	2.8189	2.8202

Sol: Here interval of differencing is unequal. The Lagrange's interpolation formula can be used. The values of arguments and the corresponding entries in usual notations are.

$x:$	$a_0 = 654$	$a_1 = 658$	$a_2 = 659$	$a_3 = 661$
$f(x):$	2.8156	2.8182	2.8189	2.8202

Taking x = 656 in Lagrange's formula and using the above table we get

$$f_{656} = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156 + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189 + \frac{(656-654)(656-659)(656-669)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8189 + \frac{(2)(-3)(-5)}{(4)(-1)(-3)} \times 2.8182 + \frac{(2)(-2)(-5)}{(5)(1)(-2)} \times 2.8189$$

$$+ \frac{(2)(-2)(-3)}{(7)(3)(2)} \times 2.8202$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058 = 2.8168$$

15.3 INVERSE INTERPOLATION

Inverse interpolation: technique of finding the value of x for the given value of $y = f(x)$ is termed as inverse interpolation.

Inverse interpolation using Lagrange's formula:

Illustration: The values of x and   are given below. Find x when

x	:	5	6	9	11
$y = f(x)$	:	12	13	14	16

Solution: In usual notation we have

x :	$a_0 = 5$	$a_1 = 6$	$a_2 = 6$	$a_3 = 11$
$y = f(x):$	12	13	14	16

By using Lagrange's formula we get

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5 + \frac{(15-12)(15-14)(15-16)}{(13-14)(13-14)(13-16)} \times 6$$
$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$= \frac{5}{6} - 6 + \frac{27}{2} + \frac{11}{4} = 11.5$$

15.3 SUMMARY AND FURTHER SUGGESTED READING

The main objectives of studying this lesson was to understand Newtons Forward and backward interpolation formulae for equal as well as unequal interval of differencing and to know the basic concept of divided differences we have also learnt about the Lagrange's methods of interpolation for divided differences. Below given are the major findings of this lesson

- Newton's Divided Difference Formula

$$f(x) = f(a) + (x-a) f(a,b) + (x-a)(x-b) f(a,b,c) + (x-a)(x-b)(x-c) f(x,a,b,c)$$
$$+ (x-a)(x-b)(x-c)(x-d) f(x,a,b,c,d) + ...$$

- Lagrange's Interpolation Formula

$$f(x) = \frac{(x-a_1)(x-a_2)........(x-a_{n.})}{(a_0-a_1)(a_0-a_2).....(a_0-a_n)} f(a_0) + \frac{(x-a_0)(x-a_2)........(x-a_{n.})}{(a_1-x)(a_1-a_0)(a_1-a_2)(a_1-a_n)} f(a_1)$$

$$+ ....+ \frac{(x-a_0)(x-a_1)(x-a_2)........(x-a_{n-1.})}{(a_n-a_0)(a_n-a_1)........(a_n-a_{n-1.})} f(a_n)$$

If $f(x)$ is a polynomial of nth degree then rth order divided difference is zero for , $r > n$ and the series defined above terminates after (n + 1) terms

- $The\ rth\ order\ divided\ difference = f(a,b,c,d,.....,(r+1) terms) = \frac{1}{r!h^r} \Delta^r f(a)$

FURTHER SUGGESTED READING

1. Goon, Gupta, Das Gupta (1987) : Fundamentals of Statistics, Vol. 1. World Press, Calcutta.

2. Hilderbrand, F. B. IntroductiOn to Numerical Analysis. Tata McGraw Hill, 1974.

15.3 SELF ASSESSMENT QUESTIONS

1. State Newton's forward and backward interpolation formula use it to obtain $\sqrt{5.5}$ given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, , $\sqrt{7} = 2.646$ and , $\sqrt{8} = 2.828$.

2. The values of x and $y = f(x)$ are given below. Find x when $f(x) = 14$

x :	1	3	5	7
$y = f(x)$:	11	13	15	16

16.1 OBJECTIVES

The objectives of this lesson are to

- understand meaning of numerical differentiation and integration
- Understand the concepts of numerical differentiation and numerical integration.
- Understand the basic terminology and application numerical integration

16.2 INTRODUCTION TO NUMERICAL INTEGRATION AND DIFFERENTIATION

The basic idea involved in the technique of numerical differentiation is to approximate the given function $y = f(x)$ over a short interval of x by a suitable polynomial interpolation formula $\phi(x)$ and then to differentiate that formula rather than the original function.

So we can obtain an estimate of the value of the derivative of a function $f(x)$ at a given value of x even though the algebraic form of $f(x)$ is not known. In order to carry out this we require a table of values of $f(x)$.

The process of calculating the derivative of a function, with the help of the approximating interpolation formula and given a set of values of the function, is known as numerical differentiation.

The problem is solved by selecting an appropriate interpolation formula and then differentiating it term by term as many times as is desired. If the given set of values of $f(x)$ are at equidistant values of x, then we choose an interpolation formula using differences. If moreover, the derivative required is at the beginning (end or central part) of the tabulated values, then we select Newton's forward (Newton's backward or Central difference) formula. Otherwise, we use Lagrange's formula or a divided difference formula.

Let us consider some simple approximate methods of finding the value of a definite integral from a given set of numerical values of the integrand. This process is known as mechanical quadrature when the integrand is a function of single variable.

We replace the integrand by a suitable interpolation formula; usually one involving differences, and then integrates it term by term between the desired limits. We can get different quadrature formulae, as they are called, by replacing the integrand by different interpolation formula or retaining, in the same formula, terms up to different orders of difference. We shall obtain below some quadrature formulae by integrating Newton's forward formula.

16.3 GENERAL QUADRATURE FORMULA.

Let us consider the function $y = f(x)$

The given set of values of $f(x)$ are at equidistant values of x, then we choose an interpolation formula using differences.

In Newton's formula, If we take $u = \frac{(x - a_0)}{h}$ so that $dx = hdu$ and let the limits of integration for x are a_0 *and* $a_0 + nh$ so that when $x \to a_0, \; u \to 0$ and when $x \to a_0 + nh, \quad u \to n$. Hence the limits in the terms of u will be 0 and n. Hence

$$\int_{a_0}^{a_0+nh} f(x)dx = h\int_{a_0}^{a_0+nh}\left[f(a_0) + u\Delta f(a_0) + \binom{u}{2}\Delta^2 f(a_0) + \binom{u}{3}\Delta^3 f(a_0) + \ldots\ldots \right]$$

$$= h\left[nf(a_0) + \frac{n^2}{2}\Delta f(a_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(a_0)}{2!}\right.$$

$$+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 f(a_0)}{3!}+.....\Bigg] \qquad ....(1)$$

Here (1) represents the general formula from which some particular formulae for numerical integration can be deducted.

16.2 TRAPEZOIDAL RULE

Here we assume that the integrand is such that it can be well represented by a straight line in an interval of width h. That means *f(x)* can be replaced by a first degree polynomial or equivalently, $\Delta f(x)$ can be regarded as a constant. Accordingly, putting in (1) n = 1 and neglecting differences of all orders higher than the first, we get

$$\int_{a_0}^{a_1} f(x)dx = h\left[f(a_0)+\frac{\Delta f(a_0)}{2}\right]$$

$$=\frac{h}{2}\left[f(a_0)+f(a_1)\right]$$

Simi1rly,

$$\int_{a_1}^{a_2} f(x)dx = \frac{h}{2}\left[f(a_1)+f(a_2)\right]$$

....

....

$$\int_{a_{n-1}}^{a_n} f(x)dx = \frac{h}{2}\left[f(a_{n-1})+f(a_n)\right]$$

Adding all these integrals we get,

$$\int_{a_0}^{a_n} f(x)dx = \int_{a_0}^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + ... + \int_{a_{n-1}}^{a_n} f(x)dx$$

$$= \frac{h}{2}\left[f(a_0) + f(a_1)\right] + \frac{h}{2}\left[f(a_1) + f(a_2)\right] + ... + \frac{h}{2}\left[f(a_{n-1}) + f(a_n)\right]$$

$$= \frac{h}{2}\left[f(a_0) + 2f(a_1) + 2f(a_2) + ... + 2f(a_{n-1}) + f(a_n)\right]$$

$$= h\left[\frac{f(a_0) + f(a_n)}{2} + \left\{f(a_1) + 2f(a_2) + ... + 2f(a_{n-1})\right\}\right]$$

$$= h\left[\textit{Mean of extreme terms} + \textit{sum of intermediate terms}\right]$$

This is known as the trapezoidal rule. It is useful where Ii is small, for any small segment of a smooth curve can be approximated by a straight line.

Geometrically, this rule means that we are replacing the graph $y = f(x)$ between a_0 and $a_0 + nh$ by a segment of straight lines and then approximating the area under the curve by that of a polygon.

3.2 SIMPSON'S ONE THIRD RULE

The general quadrature formula from which some particular formulae for numerical integration can be deducted is given by

$$\int_{a_0}^{a_0+nh} f(x)dx = h\left[nf(a_0) + \frac{n^2}{2}\Delta f(a_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(a_0)}{2!}\right.$$

$$\left. + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(a_0)}{3!} + ...\right]$$

……..(1)

If we assume that the integrand is such that it can be replaced by a second- degree polynomial over any interval of width 2h. Accordingly, we put in (1) n =2 and ignore differences of all orders above the second. We have then

$$\int_{a_0}^{a_2} f(x)dx = h\left[2f(a_0) + 2\Delta f(a_0) + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 f(a_0)}{2!}\right]$$

$$= \frac{h}{3}\left[f(a_0) + 4f(a_1) + f(a_2)\right]$$

Similarly

$$\int_{a_2}^{a_4} f(x)dx = \frac{h}{3}\left[f(a_2) + 4f(a_3) + f(a_4)\right]$$

$$\ldots \qquad \ldots \qquad \ldots.$$

$$\ldots \qquad \ldots \qquad \ldots.$$

$$\int_{a_{n-2}}^{a_n} f(x)dx = \frac{h}{3}\left[f(a_{n-2}) + 4f(a_{n-1}) + f(a_n)\right]$$

Finally, we have (assuming n is even) and adding these integrals we get.

$$\int_{a_0}^{a_n} f(x)dx = \int_{a_0}^{a_2} f(x)dx + \int_{a_2}^{a_4} f(x)dx + \ldots\ldots. + \int_{a_{n-2}}^{a_n} f(x)dx$$

$$= \frac{h}{3}\left[f(a_0) + 4f(a_1) + f(a_2)\right] + \frac{h}{3}\left[f(a_2) + 4f(a_3) + f(a_4)\right] + \ldots.$$

$$+ \frac{h}{3}\left[f(a_{n-2}) + 4f(a_{n-1}) + f(a_n)\right]$$

$$= \frac{h}{3}\Big[f(a_0) + 4\{f(a_1) + f(a_3) + \ldots + f(a_{n-1})\}$$

$$+ 2\{f(a_2) + 4f(a_4) + f(a_n)\}\Big]$$

This is termed as Simpson's one third rule

The rule is very simple, accurate and one of the most important of all quadrature formulae. In this case, we have assume intervals are divided into even number of sub intervals and geometrically, this means that we have replaced the given function by n/2 arcs of second degree pllynomilas.

16.2 SIMPSON'S THREE EIGHTH RULE

The general quadrature formula from which some particular formulae for numerical integration can be deducted is given by

$$\int_{a_0}^{a_0+nh} f(x)dx = h\left[nf(a_0) + \frac{n^2}{2}\Delta f(a_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(a_0)}{2!} \right.$$

$$\left. + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(a_0)}{3!} + \ldots.. \right] \qquad \ldots\ldots..(1)$$

If we assume that the integrand is such that it can be replaced by a third- degree polynomial over any interval of width 3h. Accordingly, we put in (1) n =3 and ignore differences of all orders above the third. Similarly proceeding as in the case of Simpson one third rule (Left for Students).

16.2 WDDLE'S RULE

Here we replace the integrand by a sixth degree polynomial over any interval of width 6h and we ignore differences above the sixth in (1) in the general quadrature formula. We have then, after some simplifications,

$$\int_{a_0}^{a_6} f(x)dx = h\left[6 f(a_0) + 18\Delta f(a_0) + 27\Delta^2 f(a_0) + 24\Delta^3 f(a_0) + \frac{123}{10}\Delta^4 f(a_0) \right.$$

$$\left. + \frac{33}{10}\Delta^5 f(a_0) + \frac{41}{140}\Delta^6 f(a_0)\right]$$

The coefficient of $\Delta^6 f(a_0)$ differs from 3/10 by a small fraction, $1/140$. Making this change in the coefficient of $\Delta^6 f(a_0)$, which will be negligible is small, we get

$$\int_{a_0}^{a_6} f(x)dx = \frac{3h}{10}\left[f(a_0) + 5f(a_1) + f(a_2) + 6f(a_3) + f(a_4) + 5f(a_5) + f(a_6)\right]$$

Similarly, we have

$$\int_{a_6}^{a_{12}} f(x)dx = \frac{3h}{10}\left[f(a_6)+5f(a_7)+f(a_8)+6f(a_9)+f(a_{10})+5f(a_{11})+f(a_{12})\right]$$

Proceeding in a similar manner for n integrals (assuming n is a multiple of 6) and finally, adding these integrals we have

$$\int_{a_0}^{a_n} f(x)dx = \int_{a_0}^{a_6} f(x)dx + \int_{a_6}^{a_{12}} f(x)dx + ...+ \int_{a_{n-6}}^{a_n} f(x)dx$$

$$= \frac{3h}{10}\left[f(a_0)+5f(a_1)+f(a_2)+6f(a_3)+f(a_4)\right.$$

$$+5f(a_5)+2f(a_6)+.......+f(a_{n-6})+5f(a_{n-5})$$

$$\left.+f(a_{n-4})+6f(a_{n-3})+f(a_{n-2})+5f(a_{n-1})+f(a_n)\right]$$

This is Weddle's rule.

Illustrations:

1. Calculate the value of the definite integral $\int \frac{1}{x}$ correct to five places of decimals using the trapezoidal, Simpson one third and Weddle's rule.

Solution: Divide the interval (1,2) into six equal parts each of width $\frac{1}{6}$. The values of the function $y = \frac{1}{x}$ are next tabulated for each of the seven boundaries:

x	$y = \frac{1}{x}$
1	1.00000
7/6	0.857143
8/6	0.750000
9/6	0.666667
10/6	0.600000
11/6	0.545455
12/6	0.500000

(a) By using the trapezoidal rule, the integral is evaluated as .

$$\int_{a_0}^{a_n} f(x)dx = \int_1^2 f(x)dx = \frac{1}{12}[1.50000 + 2 \times 3.419265]$$

$$=0.69488$$

(b) Simpson sine third rule gives

$$I_{1/3} = \int_1^2 f(x)dx = \frac{1}{18}[1.50000 + 4 \times 2.069265 + 2 \times 1.350000]$$

$$= 0.69137$$

(c) By Weddle's rule, we have for the integral the value

$$I_{weddle} = \frac{1}{20}[2.85 + 5 \times 1.408598 + 6 \times 0.66667]$$

$$= 0.69315$$

2. Evaluate Log_e^7 by Simpson's one-third rule.

Solution : Let us consider the integral,

$$\int_0^6 \frac{1}{1+x} dx = \left| Log_e(1+x) \right|_0^6 = Log_e^7$$

$$= \int_0^6 f(x)dx, \qquad where \quad f(x) = \frac{1}{1+x}$$

Now $f(a_0) = 1$, $f(a_2) = \frac{1}{2}$, $f(a_3) = \frac{1}{4}$, $f(a_4) = \frac{1}{5}$

$f(a_5) = \frac{1}{6}$, $f(a_6) = \frac{1}{7}$

Now by using Simpson's one-third rule for 7 ordinates at equal intervals of unity (h=1), we get

$$\int_{a_0}^{a_0+nh} f(x)dx = \int_0^6 \frac{1}{1+x}dx = \frac{1}{3}\left[\left(1+\frac{1}{7}\right)+4\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)+2\left(\frac{1}{3}+\frac{1}{5}\right)\right]$$

$$= 1.9588$$

1. If third differences are constant, prove that

$$\int_{-1}^{1} f(x)dx = \frac{2}{3}\left[f(0) + f\left(1/\sqrt{2}\right) + f\left(-1/\sqrt{2}\right)\right]$$

Solution. Since third differences are constant, f(x) can be taken as a 3^{rd} degree polynomial

$$f(x) = a + bx + cx^2 + dx^3 \qquad \ldots\ldots\ldots..(1)$$

$$L.H.S = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} f(a+bx+cx^2+dx^3)dx$$

$$= 2\left(a+\frac{3}{3}\right)$$

$$R.H.S = \frac{2}{3}\left[a + \left(a + b/\sqrt{2} + c/2 + \frac{d}{2\sqrt{2}}\right) + \left(a - b/\sqrt{2} + c/2 - \frac{d}{2\sqrt{2}}\right)\right]$$

$$= \frac{2}{3}\left[a + 2\left(a+\frac{c}{2}\right)\right] = \frac{2}{3}(3a+c)$$

$$= 2\left(a+\frac{c}{3}\right)$$

$$= L.H.S$$

16.2 SUMMARY AND FURTHER SUGGESTED READING

The objectives of this lesson were to introduce the concept of numerical differentiation and numerical integration and to introduce some important rules along with their applications with help of some illustrations. Below given are the main results of this lesson

- General Quadrature formula

$$\int_{a_0}^{a_0+nh} f(x)dx = h\left[nf(a_0) + \frac{n^2}{2}\Delta f(a_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(a_0)}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(a_0)}{3!} + \ldots.. \right]$$

- Trapezoidal Rule

$$\int_{a_0}^{a_n} f(x)dx = h\left[\frac{f(a_0) + f(a_n)}{2} + \left\{f(a_1) + 2f(a_2) + \ldots\ldots\ldots + 2f(a_{n-1})\right\}\right]$$

$$= h[Mean\ of\ extreme\ terms + sum\ of\ intermediate\ terms]$$

- Simpson's one third rule

$$\int_{a_0}^{a_n} f(x)dx = \frac{h}{3}\left[f(a_0) + 4\{f(a_1) + f(a_3) + \ldots + f(a_{n-1})\} + 2\{f(a_2) + 4f(a_4) + f(a_n)\}\right]$$

- Weddle's rule

$$\int_{a_0}^{a_n} f(x)dx = \int_{a_0}^{a_6} f(x)dx + \int_{a_6}^{a_{12}} f(x)dx + \ldots\ldots. + \int_{a_{n-6}}^{a_n} f(x)dx$$

$$= \frac{3h}{10}[f(a_0) + 5f(a_1) + f(a_2) + 6f(a_3) + f(a_4)$$

$$+ 5f(a_5) + 2f(a_6) + \ldots\ldots. + f(a_{n-6}) + 5f(a_{n-5})$$

$$+ f(a_{n-4}) + 6f(a_{n-3}) + f(a_{n-2}) + 5f(a_{n-1}) + f(a_n)]$$

FURTHER SUGGESTED READINGS

1. Goon, Gupta, Das Gupta (1987) : Fundamentals of Statistics, Vol. 1. World Press, Calcutta.
2. Hilderbrand, F. B. IntroductiOn to Numerical Analysis. Tata McGraw Hill, 1974.

16.2 SELF ASSESSMENT QUESTIONS

1 Describe in brief the (a) Trapezoidal rule and(b) Simpson's one-third rule for

numica1 integration.

2. Use Simpson's one-third rule to estimate the value of $\int_1^5 f(x)dx$, given that

x:	10	20	30	40	50
f(x)	10	50	70	80	90

3. Compute the value of the integral $\int_4^{5.2} Logxdx$ by any suitable method of numerical integration.

4. Use Simpson's one-third rule to estimate the approximate value of Log_e^2 , from the integral

$\int_0^1 \frac{1}{x} dx$ by taking five ordinates.

17.1 OBJECTIVES

After successful completion of this lesson students will be able to:

- Know about methods of finding the roots of an equation.
- Know about the numerical solutions of the non linear equations
- Solve non-linear equations using iterative techniques.
- Know about convergence of some of these methods.

17.2 AN INTRODUCTION TO NUMERICAL SOLUTION OF EQUATIONS

Numerical Solutions of equations: Here we are going to discuss some methods of finding the roots of an equation to any desired degree of accuracy, when the coefficients of the equation are pure numbers. It is true that most of the methods used here are also applicable to more than one variable, we shall consider the case of one variable only, and moreover we shall concentrate on the determination of real roots only and so on. We repeat our process till we get the root corrected upto desired number of decimal points for which we need to start with methods of finding approximate values of roots.

Let $f(x)=0$ be the equation whose roots we want to find then the graph of $y = f(x)$ will cross the x-axis at the point whose abscissa are the roots. Approximate values of the roots are therefore can be found we need only that part of it where it crosses the x-axis.

Another method is based on the fact that if $f(x)$ is continuous in an interval containing the root and if in that internal we find that $f(a)$ and $f(b)$ are of opposite signs then thee will at least one real root between a and b. for convergence of the approximate values of a root to true value , it is necessary that a and b should be close to each other.

In the forthcoming sections we are going to discuss various methods of determining the real roots of an equation.

17.3 METHOD OF FALSE POSITION

Method of False position

Method of "false position" or "regula fasi" is one of the oldest methods of determining the real root of a numerical equation.

Suppose that the desired roots lies between x_1 *and* x_2 which are as close as possible and further let $f(x_1)$ *and* $f(x_2)$ have opposite sign.

Assuming the part of curve $y = f(x)$ between to be smooth, we can approximate this part of the curve by the straight line. In other words, we perform linear interpolation to find the roots of and we get

$$\frac{-f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Or $$x = x_1 - \frac{(x_2 - x_2) f(x_1)}{f(x_2) - f(x_1)}$$

Or $$x = x_1 + \frac{(x_2 - x_2)|f(x_1)|}{|f(x_2)| + |f(x_1)|} \qquad \text{……………….(1)}$$

This vale of x is however not the true value of the root . This is only a better approximation to the true root than either or , we repeat this process till we get the root correct upto desired number of places

17.4 NEWTON-RAPSON METHOD

Newton-Raphson Method

By using this method the real roots of $f(x) = 0$ can be computed rapidly if the derivative of $f(x)$ is a simple expression and is easily derivable.

Let x_0 be an approximate value of the root and h is the correction to be applied, so that $x_0 + h$ is the corrected value of the root.

Then taking $f(x_0 + h) = 0$ and expanding $f(x_0 + h) = 0$ by Taylor's theorem, we get.

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0 + \theta h) = 0, \qquad 0 < \theta < 1$$

If x_0 is a quite good approximation to the unknown root, h will be small and we can then neglect the term involving h^2. Thus we have the relation

$$f(x_0) + hf'(x_0) = 0,$$

This gives the approximation to h as

$$h_1 = \frac{-f(x_0)}{f'(x_0)} \qquad \text{......................(2)}$$

The improved value of the root is then given by

$$x^{(1)} = x_0 + h_1,$$

the other approximations are given by

$$x^{(2)} = x^{(1)} + h_2 \quad \text{where } h_2 = \frac{-f\left(x^{(1)}\right)}{f'\left(x^{(1)}\right)},$$

$$x^{(3)} = x^{(2)} + h_3 \quad \text{where } h_3 = \frac{-f\left(x^{(2)}\right)}{f'\left(x^{(2)}\right)}, \text{ and so on.}$$

This process is repeated till we get the root correct up to desired number of places (desired accuracy)

Equation (2) is of fundamental importance in this method. The larger the value of $f'(x)$ near the root, more rapid will be the convergence of $x^{(2)}, x^{(1)}$,.................. to the actual root. The small value of near the root slow down the convergence of

$$x^{(2)}, x^{(1)}, \ldots\ldots\ldots\ldots$$

If this method will fail in the neighbourhood of the root.

Illustration: find the real roots of $2x - \log x = 7$ correct to five decimal places

Solution: Here $f(x) = 2x - \log x = 7$ and it is found that $f(3)$ *and* $f(4)$ are of opposite signs. So ther is a real root of the equation $f(x) = 0$ between 3 and 4 .

Newton-Raphson Method

$$f(x) = 2x - \log x = 7$$

So that $f'(x) = 2 - \frac{\log_e}{x}$

It may be easily seen from the graph that an approximate value of the root is 3.7. Hence

$$h_1 = \frac{-f(x_0)}{f'(x_0)} = \frac{7 + 0.5682017 - 7.4}{2 - 0.1146196} = \frac{0.1682017}{1.882633} = 0.08934$$

$$x^{(1)} = x_0 + h_1 = 307 + 0.08934 = 3.789$$

$$h_2 = \frac{-f\left(x^{(1)}\right)}{f'\left(x^{(1)}\right)} = \frac{7 + 0.5785246 - 7.578}{2 - 0.1146196} = \frac{0.0005246}{1.8853804} = 0.002782$$

$$x^{(2)} = x^{(1)} + h_2 = 3.789 + 0.002782 = 3.7892$$

$$h_3 = \frac{-f\left(x^{(2)}\right)}{f'\left(x^{(2)}\right)} = \frac{7 + 0.5785475 - 7.5784}{2 - 0.1146125} = \frac{0.0001475}{1.8853875} = 0.000078$$

$$x^{(3)} = x^{(2)} + h_3 = 3.789278$$

So the root correct upto five decimal places is

Method of False position

	x	$f(x)$
Ist approx.	3	$-1.47 \quad x = 3 + \dfrac{1\times 1.47}{1.86} = 3 + 0.7903$
	4	0.39
	1	$1.86 \quad x^{(1)} = 3.7$
2nd approx.	3.7	$-0.1682 \quad x = 3.7 + \dfrac{0.1\times 0.1682}{0.1885}$
	3.8	$0.0203 = 3.7 + 0.08923$
	0.1	$0.1885\, x^{(2)} = 3.78.$
3rd approx.	3.78	$-0.1682 \quad x = 3.78 + \dfrac{0.1\times 0.01749}{0.1885}$
	3.79	$0.00136 = 3.78 + 0.009288$
	.01	$0.1885\, x^{(3)} = 3.7892$
4th approx.	3.7892	$-0.0001475 \quad x = 3.7892 + \dfrac{0.0001\times 0.0001475}{0.0001885}$
	3.7893	$0.0000410 = 3.7892 + 0.00007824$
	0.0001	$0.0001885 \quad x^{(4)} = 3.789278$

So the required root is 3.789278 corrected to five decimal places.

17.5 METHOD OF ITERATION

Method of iteration

In situations where the numerical equation $f(x) = 0$ can be rewritten as

$$x = \phi(x) \qquad \ldots\ldots\ldots..(3)$$

The real roots can be determined by a process known as iteration or successive approximation. Here we start the approximation with an approximate value x_0, substituting it on the right hand side of the equation (3), we get an improved value of the root $x^{(1)}$ given by

$$x^{(1)} = \phi(x_0)$$

Again if we put $x^{(1)}$ for x on the right hand side of the equation (3), we get the second approximation as

$$x^{(2)} = \phi(x^{(1)})$$

$$x^{(3)} = x^{(2)} + h_3 \quad \text{where } h_3 = \frac{-f\left(x^{(2)}\right)}{f'\left(x^{(2)}\right)}, \text{ and so on.}$$

This process is repeated untill we get the root correct upto desired number of places. This method is used only when

$$|\phi'(x)| < 1$$

Smaller the derivative the more rapid will be the convergence of the approximate roots to the real roots.

Illustration: Find by the method of iteration the positive roots of the equation $e^x = 1 + 2x$ corrected to four decimal places.

Solution: here $e^x = 1 + 2x$ **or**

$$x = \ln(1 + 2x)$$
$$= \log(1 + 2x)/\log e$$

Taking $f(x) = x - \log(1 + 2x)/\log e$ and forming a set of values of $f(x)$ for different values of x , it is found that $f(1.25)$ *and* $f(1.26)$ are oif opposite signs.So a positive root lies between 1.25 and 1.26. Thus we begin the process of iteration with x=1.25. as the starting value.

x	$\phi(x) = \log(1+2x)/\log e$
1.250000	1.252760
1.252760	1.254336
1.254336	1.255235
1.255235	1.255747
1.255747	1.256038
1.256038	1.256205
1.256205	1.256299
1.256299	1.256353
1.256353	1.256384
1.256384	1.256402
1.256402	1.246412
1.256412	1.256418
1.256418	1.256424
1.256426	1.256427

So the root is 1.2564, correct decimal places.

17.6 CONVERGENCE OF THE ITERATION AND NEWTON-RAPSON METHOD

METHOD

Convergence of Method of iteration and Newton-Raphson Method

Let us consider the condition under which the iteration method will converge, i.e., the condition under which the successive approximations x_0, $x^{(1)}$ $x^{(2)}$,………… will tend to the true value of the root. The true value of the root satisfies the equation.

$$x = \phi(x)$$

And the first approximation is obtained as

$$x^{(1)} = \phi(x_0)$$

Subtracting , we get

$$x - x^{(1)} = \phi(x) - \phi(x_0)$$

By the mean value theorem

$$\phi(x) - \phi(x_0) = (x - x_0)\phi'(z_0)$$

Where z_0 is a point in the interval (x_0, x) or (x, x_0). Thus we have

$$x - x^{(1)} = (x - x_0)\phi'(z_0)$$

Similar equations will hold for other approximations

$$x - x^{(2)} = (x - x^{(1)})\phi'(z_1),$$

$$x - x^{(3)} = (x - x^{(2)})\phi'(z_2),$$

……………………………..
……………………………..
……………………………..

$$x - x^{(n)} = (x - x^{(n-1)})\phi'(z_{n-1}),$$

Multiplying together the n equations by element to element and dividing by the common factor $(x - x^{(1)})(x - x^{(2)})$.......$(x - x^{(n)})$ we get

$$x - x^{(n)} = (x - x_0)\prod_{i=0}^{n-1}\phi'(z_i)$$

Now if the value m (say) of $|\phi'(z_i)|$ is less than 1 in the interval (x_0, x) or (x, x_0), so that $|\phi'(z_i)|,\ m < 1$ for each I, we have

$$\left|x - x^{(n)}\right| \le m^n |x - x_0| \qquad \text{………..(4)}$$

Thus the error after n repetitions of the process can be made as small as we please by increasing n suitably since the right hand side of (4) depends upon m^n which approaches to zero as n increases .

Thus the condition for convergence of iteration method is that $|\phi'(x)| < 1$ is in the neighbourhood of the desired root , where $f(x)$ is the function occurring in the (3) . The smaller the value of $|f'(x)|$ the more rapid is the convergence.

CONVERGENCE OF NEWTON-RAPHSON METHOD

The Newton-Raphson method can also be considered as iteration method. The nth approximation in this method can by given by

$$x^{(n)} = x^{(n-1)} - \frac{f\left(x^{(n-1)}\right)}{f'\left(x^{(n-1)}\right)}$$

Which may be written in the form

$$x = \phi(x),$$

With $$\phi(x) = x - \frac{f(x)}{f'(x)}$$

The iteration method, we know that Newton-Raphson method will converge if,

$$\left|\frac{d}{dx}\left[x - \frac{f(x)}{f'(x)}\right]\right| < 1$$

i.e., if

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$ is in the neighbourhood of the required root.

SUMMARY AND FURTHER SUGGESTED READING

In the present lesson our focus was on the methods of finding the roots of an equation and numerical solutions of the non linear equations by using iterative techniques, Newton-

Raphson method and method of false position etc. The real roots can be determined by a process known as iteration or successive approximation**.** Here we repeat our process till we get the root corrected upto desired number of decimal points for which we need to start with methods of finding approximate values of roots.

By using Newton-Raphson Method the real roots of $f(x)=0$ can be computed rapidly if the derivative of $f(x)$ is a simple expression and is easily derivable. Method of "false position" or "regula fasi" is one of the oldest methods of determining the real root of a numerical equation to any desired degree of accuracy

FURTHER SUGGESTED READINGS:

1. Goon Gupta, Das Gupta (1987): Fundamentals of Statistics, Vol.I, World Press Calcutta.

2. Hilderbrand,F.B: Introduction to Numerical Analysis, Tata McGraw Hill, 1974.

SELF ASSESSMENT QUESTIONS

Exercise: Find the largest positive root of the equation

$$x^3 - 4.876x^2 + 7.50 = 0$$

Exercise: Find the smallest positive real root of the equation

$$xe^x - 2 = 0$$

By using the method of false position to four significant places.

Exercise: By using the Newtoin-Raphson method to five significant places the real root of the equation

$\sin x - \dfrac{x+1}{x-1} = 0$ (an approximate value of the root from graphs of $y = \sin x$

and $y = \dfrac{x+1}{x-1}$ is known to be -0.40

Exercise: Find the positive root of the equation $x^x + 5x - 1000 = 0$ correct to four decimal places.

UNIT-V

Lesson - 18

Lesson - 19

Lesson - 20

18.1 OBJECTIVES

After studying this lesson students should be able to

- Define a Linear Programming Problem.
- Formulate a Linear Programming Problem mathematically.
- Understand basic terminology for solution of Linear Programming problem.

18.2 AN INTRODUCTION TO LINEAR PROGRAMMING PROBLEMS

Linear programming, one of the important techniques of operations research, has been applied to a wide range of business problems. This technique is useful in solving decision making problems which involve maximizing a linear objective function subject to a set of linear constraints.

The important applications of this technique are in the following areas:

- Selection of a product mix which maximizes the profits of the firm subject to several production, material, marketing, personnel and financial constraints.
- Determination of the capital budget which maximizes the net present value of the firm subject to several financial, managerial, environmental, and other constraints.
- Choice of mixing short-term financing which minimizes the cost subject to certain funding constraints.

Linear programming is an optimization technique for finding an optimal (maximum or minimum) value of a function, called objective function, of several independent variables, the variables being subject to various restrictions (or constraints) expressed as equations or inequalities. The term ‘Linear’ indicates that the function to be maximized or minimized is linear in nature and that the corresponding constraints be represented by a system of linear inequalities or linear equations involving the variables.

In general, programming problems deal with determining optimal allocations of limited resources to meet given objectives. The resources may be in the form of men, materials and machines etc. and the objective may be to yield one or more-products. There are certain restrictions on the total amount of each resource available, and on the quantity or quality of each product made. Out of all permissible allocations of resources, one has to find the one which optimizes (maximize or minimize) the total profit or cost.

In this lesson, we shall discuss the General Linear Programming problems and how they are mathematically formulated. Also we shall give some basic terms useful for solution of linear programming problems (LPP).

18.3 ASSUMPTIONS AND ADVANTAGES OF LPP

Advantages of Linear Programming.

- It improves the quality of ions. It indicates how the available resources can be used in a best way.
- It helps in attaining the optimum use of productive resources and man-power.
- It also reflects the drawbacks of the production process.
- Linear programming method allows the modification of its mathematical solutions.
- It improves the knowledge and skill of tomorrow’s executives.

Limitations of linear programming. Besides the wide field of application, in practical situations, the linear programming technique has certain limitations as follows

(i) The L.P. technique is applicable to that class of applications only in which the objective function and the constraints both involves linear functions. But in many practical problems, it is not possible to express both the objective function and the constraints in linear form. Then this technique can not be used.

(ii) If the problem involves large number of limitations and constraints and decision variables, then computations become tedious even on large digital computers.

(iii) Many times the decision variables may be required to have only integral values, while as L.P. technique allows the solution variables to attain any value

(iv) In a L.P.P. all the coefficients in the constraints and the objective functions must be completely known constants. Contrary if some one of the coefficients is variable or random variable with certain probability distribution, then LLP. technique can not be used to solve the prob1em .

18.4 GENERAL LINEAR PROGRAMMING PROBLEM

A Linear Programming Problem (L.P.P.) includes a set of simultaneous linear equations which represent the conditions of the problem and a linear function which expresses the objective function of the problem. The Linear Function which is to be optimized is called the objective function and the conditions of the problem expressed as simultaneous linear equations (or inequalities) are referred as constraints.

A general Linear Programming problem can be stated as follows:

Find $x_1, x_2, x_3 .... x_n$ which optimizes the linear function

$$Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + ... + c_n x_n \qquad ......(1)$$

Subject to the constraints

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{in}x_n (\le = \ge) b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + a_{2n}x_n (\le = \ge) b_2 \\
------------------------ \\
------------------------ \\
a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n (\le = \ge) b_i \\
------------------------ \\
------------------------ \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n (\le = \ge) b_m
\end{array}\right] \qquad ....(2)$$

and non-negative restrictions.

$x \geq 0, \qquad j = 1,2,,............n;$

where a_{ij}'s , b_j's and c_j's are constants and x_j's are variables(decision variables).

The function given by (1) is called the objective function and the conditions described in (2) are termed as constraints of L.P.P

Here in the set of conditions $(\leq = \geq)$ means that any of the three signs may be there. Also optimize means either maximize or minimize. The linear function which is to be optimized is called the objective function. the conditions are referred as constraints. Any problem which can be formulated in the above form is called a L.P.P. By finding a solution to (1 and 2), we mean to find the non-negative values of variables x_1, x_2, x_3, x_n which optimize z and satisfy all the constraints.

The above L.P.P may also be stated in matrix form as:

Optimize (Maximize or Minimize) Z=CX

Subject to

$$AX(\leq = \geq)B$$

And $X \geq 0$,

Where $A = \lfloor a_{ij} \rfloor$ is a coefficient matrix of the order $m \times n$

$C = (c_1, c_2, c_n)$ is a row vector,

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_n \end{bmatrix}$ is column vector of variables(decision variables) and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_m \end{bmatrix}$ is column vector called the requirement vector and 0 is n dimensional null column vector.

- A set of values of x_1, x_2, x_3, x_n which satisfies the set of constraints and the non negativity restrictions, is called a Feasible Solution (F.S.).

A F.S. which also optimizes the objective function is called an optimal solution.

MATHEMATICAL FORMULATION OF L.P.P

The conversion of oral description into a mathematical form in L.P.P is termed as its formulation. It is important to recognize a problem which can be handled by Linear programming and then to formulate its mathematical model.

One of the important point of the linear programming is to recognize a problem that can be handed by linear programming and then to formulate the mathematical model of it. The main steps to represent a linear program in symbols are as follows:

Step I. Identify the unknown decision variables to be determined and assign symbols to them.

Step II. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.

Step III. Identify the objective or aim and represent it also as a linear function of decision variables.

Requirements for formulating a Linear Problem. There are five basic necessary requirements for the formulation of a L.P.P .

(i) *Well defined linear objective function*. A linear objective function must be clearly defined mathematically.

(ii) *Alternative courses of action*. There must be alternative courses of action so that a problem of choosing best may arise.

(iii) *Linear constraint must be expressed mathematically*. The constraints must be .capable of being expressed mathematically in the form of linear equation or inequalities.

(iv) Variables in the problem must be interrelated, so that it may be possible to formulate mathematical relationship among them.

(v) Resources must be limited i.e. they must be finite and economically quantifiable.

Here we are giving some of the examples where the mathematical models of given problems are formulated. Formulation of models is no science but an art which will be more refined to you by practice

Illustration 1:

Let us consider a small foundry which specializes in the production of iron castings. For the sake of simplicity, assume that the foundry specializes in producing two types of castings — type A and type B. It is assumed that the foundry can sell as many units as it produces. The profit is Rs.70 and Rs.40 for each of casting A and casting B respectively. The foundry manager should decide the quantity of these castings to be produced each week so as to maximize the total profit.

Production of castings requires certain resources like raw materials, labor and foundry capacity.

The requirements and their availabilities are given in the following table:

Resources	Requirement per unit of		Availability in a week
	type A	type B.	
Raw material-I	2Kgs.	1Kg	120 Kgs
Raw material-II	0.8 Kgs	None	40 kgs
Labour	3 man-days	2 man-days	200 man days
Foundry capacity	4 units	3units	360 units

Let us formulate the problem mathematically

As the manager has to decide the number of type A and type B castings to be produced, let us define the variables:

x_1 = number of type A castings to be produced

x_2 = number of type B castings to be produced

For this production schedule, the total profit will be

$$70x_1 + 40x_2$$

This function is known as the objective function which is to be maximized. If there are no constraints, the profit can be increased to infinity. In real life, there are restrictions of different kinds. These are formulated as constraints.

Let us consider raw material-I, of which only 120 kgs are available, if x_1 of type A castings and x_2 of type B castings are produced, then the requirement of raw material-I is

$$2x_1 + 1x_2,$$

and this should be less than or equal to the available quantity of raw material-I. This can be shown by the following inequation:

$$2x_1 + 1x_2 \leq 120$$

This implies that we are interested in the values of q and q2 for which the left-hand-side value is less than or equal to the right-hand-side value of 120. Otherwise, the requirement will exceed the availability and the production of that quantity will not be feasible.

By a similar argument, we get the constraints for raw material-2, labor and foundry capacity as:

Raw material-2	:	$0.8x_1 + 0x_2 \leq 40$
Labour	:	$3x_1 + 2x_2 \leq 200$
Foundry Capacity	:	$4x_1 + 3x_2 \leq 360$

As one cannot produce negative quantities, we have the restrictions:

$$x_1 \geq 0, \qquad x_2 \geq 0$$

Putting together the above elements, the problem may be represented as:

$$\text{Maximize } Z = 70x_1 + 40x_2 \qquad \ldots\ldots(1)$$

Subject to constraints:

$$\left.\begin{aligned} 2x_1 + 1x_2 &\leq 120 \\ 0.8x_1 + 0x_2 &\leq 40 \\ 3x_1 + 2x_2 &\leq 200 \\ 4x_1 + 3x_2 &\leq 360 \end{aligned}\right\} \qquad \ldots\ldots..(2)$$

$x_1 \geq 0, \qquad x_2 \geq 0 \qquad \qquad \ldots\ldots(3)$

We have to find the values of x_1 and x_2 which will satisfy constraints (2) and (3) and at the same time maximize function (1). The function given in (1) is called an objective function. The inequalities in (2) are called constraints and the inequalities in (3) are called non-negativity restrictions or constraints. This problem cannot be solved by the calculus method because of the inequality constraints.

Illustration 2: An industry manufactures two types of lamps say A and B. Both the lamps go through two technicians, first a cutter, second a finisher.

Lamp A requires 2 hours of the cutter's time and 1 hour of the finisher's time. Lamp B requires 1 hour of the cutter's and 2 hours of the finisher's time. The cutter has 104 hours and finisher 76 hours of available time each month. Profit on one lamp A is Rs. 6.00 and on one lamp B is Rs. 11.00. Assuming that he can sell all that he produces, how many of each type lamps should he manufacture to obtain the best return?

Solution: Suppose he manufactures x lamps of A type and y lamps of B type. Then he wants to maximize the total profit given by

$Z = 6x + 11y$

Now the total cutter's time used in preparing x lamps of A and y lamps of B type is

$2x + y.$

It should be less than or equal to 104. This yield

$2x + y \leq 104$

Similarly, the finisher's time implies

$x + 2y \leq 76.$

Therefore, the decorator's problem is to find x,y which maximize,

$Z = 6x + 11y$

s.t. $\quad 2x + y \leq 104,$

$x + 2y \leq 76.$

$x, y \geq 0$.

Here x and y both are non-negative as the quantity prepared by decorator can not be negative.

18.6 BASIC TERMINOLOGY OF L.PP SOLUTION

1. Basic Solution. Consider a system $Ax=b$ of m equations in n unknowns (n>m) and $\text{Rank (A)} = \text{Rank (Ab)} = m$. Then none of the equations is redundant.

A solution obtained after setting exactly (n-m) variables to zero, provided the determinant formed by the columns associated to the remaining m variables, is non-zero, is called a Basic Solution.

Here we note that the matrix formed by the coefficients of the in variables, or say formed by the vectors associated to the basic variables, is non-singular as its determinant does not vanish. Hence, the vectors associated to the basic variables (vectors formed by the coefficients of the variable x_j is the vector associated to the x_j) are L.I. Thus a basic solution can be constructed by selecting the m linearly independent vectors out of n and setting the variables associated to the remaining n-m columns to zero. Thus if B is the matrix of the selected *m* linearly independent vectors and X_B is the vector of corresponding variables, then the solution of the resulting system (setting n-m variables to zero) is given by

$$Bx_B = b \quad or \quad x_B = B^{-1}b$$

if $B = (\alpha_1, \alpha_2, ...\alpha_m)$ and

$$x_B = [x_{B_1}.x_{B_2}.....x_{B_m}, 0, ..., 0]$$

then we get

$$x_B = x_{B_1}\alpha_1 + x_{B_2}\alpha_2 + ... + x_{B_m}\alpha_m + 0.\alpha_{m+1} + 0.\alpha_{m+2}, ... + 0.\alpha_n = b$$

Or $$x_{B_1}.x_{B_2}.....x_{B_m}\begin{bmatrix} x_{B_1} \\ x_{B2} \\ . \\ . \\ . \\ x_{B_m} \end{bmatrix} = b$$

Or $Bx_B = b$

Thus a solution in which the vectors associated to m variables are L.I and remaining n-m in variables are zero, is called a basic solution. For a solution to be basic, at least n-m variables must be zero.

Basic solutions play a very important role in the theory of simplex method. The superiority of these solutions is that they are finite in number. In m vectors out of n can be selected in

$$^n\mathrm{C}_\mathrm{m} = \frac{\mathrm{n!}}{\mathrm{m!(n-m)!}}$$

ways and hence $^n\mathrm{C}_\mathrm{m}$ is the upper bound for the number of basic solutions. Note that a B.S. corresponds to some basis. The m variable associated with the columns of the above non-singular matrix which may be different from zero are called basic variables. If any of the basic variables vanishes, the solution is called degenerate basic solution. On the other hand, if none of the basic variables vanishes, the solution is called non-degenerate basic solution. Thus a non-degenerate basic solution contains exactly m non-zero and n-m zero variables.

A necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of A x = b, is that every set of m columns of the augmented matrix $\mathrm{A_b} = [\mathrm{A, b}]$ is L.I.

Thus the basic Solutions are of two types:

(a) Non-degenerate Basic Solution — A basic solution is called non-degenerate basic solution if none of the basic variables is zero. In other words all the m basic variables are non-zero.

(b) Degenerate Basic Solution: A basic solution is called degenerate basic solution if one or more of the basic variables are zero.

2. A Feasible Solution: A feasible solution to a linear programming problem is the set of values of the variables which satisfies the set of constraints and the non negative restrictions of the problem.

3. Optimum Solution: A feasible solution to a L.P. problem is said to be optimum (or optimal) if it also optimizes the objective function Z of the problem.

4. Basic Feasible Solution: In a L.P.P a feasible solution which is also basic is called a basic feasible solution. Thus a F.S. in which n—m variables are zero and the vectors associated to the remaining m variables, called basic variables, are L.I., is called a B.F.S. Obviously a F.S. which contains more than m positive variables is not a B.F.S.

A basic feasible solution, in which at least one of the basic variables vanishes, is called a degenerate B.F.S. On the other hand if none of the basic variables vanishes, i.e., there are exactly m non-zero variables, then it is called non degenerate basic feasible solution.

ILLUSTRATION: Show that the feasible solution $x_1 = 1, x_2 = 0, x_3 = 1, \quad \text{and} \quad Z = 6$

to the system

$$x_1 + x_2 + x_3 = 2,$$

$$x_1 - x_2 + x_3 = 2,$$

$$x_1 + 3x_2 + 4x3 = z \text{ (minimize)}, \ x_{ij} \geq 0 \text{ is not basic.}$$

Sol. The given feasible solution contains only two non-zero variables i.e., x_1 and x_3. The vectors associated to these variables are α_1 and α_3.

Where $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

These vectors are linearly dependent as there exists scalars $\lambda_1 = 1 \ and \ \lambda_2 = -2$ such that

$$\lambda_1\alpha_1 + \lambda_3\alpha_3 + = 1.\alpha_1 + (-1)\alpha_3 = 0$$

Hence given feasible solution is not basic.

ILLUSTRATION: Find all the basic feasible solutions for the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

$$x_j \geq 0$$

18.7 SUMMARY AND FURTHER SUGGESTED READING

The main objectives of this lesson was to give basic understanding of linear programming problem and its mathematical formulation with some important terms involved in LPP . The main findings can be summarized as given below

- The conversion of oral description into a mathematical form in L.P.P is termed as mathematical formulation of a L.P.P
- Requirements for formulating a Linear Problem involves a *well defined linear objective function*, *alternative courses of action, mathematical expression forLinear constraint,* interrelation of variables.
- A solution obtained after setting exactly (n-m) variables to zero, provided the determinant formed by the columns associated to the remaining m variables, is non-zero, is called a Basic Solution.
- the basic Solutions are of two types: Non-degenerate Basic Solution and Degenerate Basic Solution.
- A feasible solution to a linear programming problem is the set of values of the variables which satisfies the set of constraints and the non negative restrictions of the problem.
- A feasible solution to a L.P. problem is said to be optimum (or optimal) if it also optimizes the objective function Z of the problem.

- In a L.P.P a feasible solution which is also basic is called a basic feasible solution.
- A basic feasible solution, in which at least one of the basic variables vanishes, is called a degenerate B.F.S. On the other hand if none of the basic variables vanishes, i.e., there are exactly m non-zero variables, then it is called non degenerate basic feasible solution.

FURTHER SUGGESTED READING

1. Kanti Swarup, Gupta and Man Mohan: Operations Research, Sultan Chand & Sons.
2. V.K. Kapoor (2001) Operations Research. Sultan Chand & Sons
3. R.K. Gupta Linear proramming, Krishna Prakashan Mandir Publishers, Meerut.
4. Mital and Sahni Linear Programming,Pragati Parkashan,Meerut

18.8 SELF ASSESSMENT QUESTIONS

Exercise: Define Linear Programming problem and give its advantages

Exercise: What do you meant by the mathematical formulation of a linear programming problem.

19.1 OBJECTIVES

To understand the theoretical base of linear programming it is necessary that one should have a sound understanding of convex sets and its properties. Keeping this in mind this lesson is devoted to the Convex Sets. After studying this lesson the students should be able to:

- Understand meaning of Convex Sets
- Understand the elementary theory of convex Sets
- Understand various properties of Convex Sets
- Understand relationship of Convex Sets with Linear Programming.

19.2 ELEMENTARY THEORY OF CONVEX SETS

1. **Set of Points:** Point sets are sets whose elements are points or vectors in E (n-dimensional euclidean space**).**
   - For example a linear equation in two variables x_1, x_2 such as $\alpha_1 x_1 + \alpha_2 x_2 = b$ represents a line in two dimensions. This line may be considered as a set of those points () which satisfy. This set of points can be written as
   - Consider the set of points lying inside a circle of unit radius centre at the origin, in two dimensional space. Obvious the points of this set satisfy the inequality .This set of points can be written as
2. **Hypersphere** : A hyper sphere in with centre at '' and radius is defined to be the

set of points.

$$S = \{x : x \in E^n, |X - a| = r\}$$

i.e., the equation of a hypersphere in E^n is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + ... + (x_n - a_n)^2$$

$$= \sum_{i=1}^{n} (x_i - a_i)^2 = r^2$$

where $a = (a_1, a_2 ...., a_n)$, $x = (x_1, x_2,........ x_n)$

which represent a circle in E^2 and sphere in E^3.

3. An $\in$-neighborhood : An $\in$-neighborhood about the point 'a' defined as the set of points lying inside the hypersphere with centre 'a' and radius $\in$>0; i.e. the $\in$-neighborhood about the point is the set of points

$$S = \{x :, |X - a| < \in\}$$

4. **An Interior Point :** A point 'a' is an interior point of the set S if there exists an about a' which contains only points of the set S. An interior point of S must be an element of S.

5. **Neighbourhood and boundary point:** A subset N of is said to be an of the point s.t.

   $\in$ being a small positive number.

   A point $x_0 \in E^n$ is said to be a boundary point of set $X \subseteq E^n$ if every $\in$-neighborhood of x_0 contains atleast one point not belonging to X and atleast one belonging to X.

   On the other hand, if there exists at least one of the point xo which contains only points of the set X, then it is called an interior point.

6. **An Open Set and a Closed Set**

   A set S is said to be an open set if it contains only the interior points. A set S is said to be a closed set if it contains all its boundary points.

8. **Lines:**

In , the line through the two points is defined to be set of points.

$$X = \{x : x = \lambda x_2 + (1-\lambda)x_1, \quad \textit{for all real } \lambda\},$$

9. **Line segment**. Line segment joining two points $x_1, x_2, \in E^n$ E^n is the set given by $X = \{x : x \in E^n \text{ and } x = \lambda x_2 + (1-\lambda)x_1, \; 0 \le \lambda \le 1\}$

The restriction $0 \le \lambda \le 1$ restricts the point x to lie within the line joining the points x_1 and x_2 . Line segment of x_1, x_2 is also

denoted by.

$$[x_1 : x_2]$$

If we eliminate this restriction i.e., λ is real only, then the set

$$X_L = \{x : x \in E^n, \; x = \lambda x_2 + (1-\lambda)x_1 \}$$

gives the whole line joining the two points x_1 and x_2, in E^n.

Obviously $X \subset E^n$ and $X_L \subset E^n$

10. Hyperplane

The equation of a hyperplane is

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = z \qquad \text{or} \qquad cx = z$$

It is represented by linear equation in n unknowns. Here not all $c_i \ne 0$ simultaneously. By using different valuesof c's and z, we can get different hyperplanes.

Further, a hyperplane is set of point $x^* \in E^n$ satisfying cx = z, where $c = (c_1, c_2, \dots, c_n)$. Thus the set

H = {x: c x = z }

is a hyperplane whose equation is c x = z.

In a graphical solution of a L.P.P every line is a hyperplane E^2

If z = 0, then c x = 0. This hyperplane is said to pass through origin.

19.3 SOME BASIC DEFINITIONS OF CONVEX SETS

CONVEX SET: Definition: A set C in n-dimensional space is said to be CONVEX if for any two points $x^{(1)}, x^{(2)}$ in set C, line segment $[x^{(1)} : x^{(2)}]$ joining these points is also in the set C.

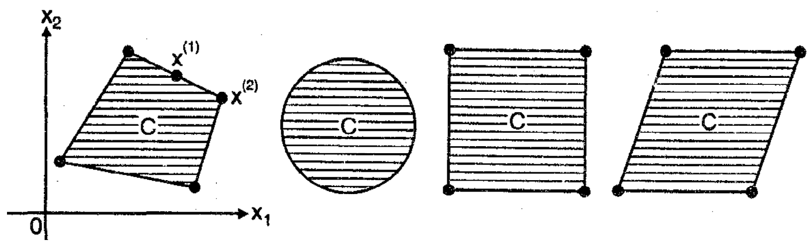
Mathematically, this definition implies that $x^{(1)}$ and $x^{(2)}$ are two distinct points in C, then every point $x = \lambda x^{(2)} + (1-\lambda)x^{(1)}$, $0 \le \lambda \le 1$, must also be in the set C.

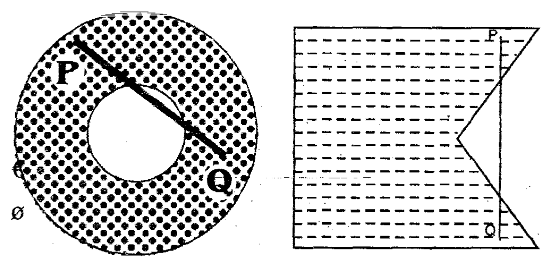
Symbolically, a subset $C \subset R_n$ is convex iff C C

It should also be noted that the set C containing only single point is convex, by convention.

The expression is called the convex combination of x, for given λ.

CONVEX SETS





Non-Convex Sets

Extreme Point of a Convex Set

A point X in a convex set C is called an extreme point if X cannot be expressed as a convex combination of any two distinct points X_1 and X_2 in C.

Obviously, an extreme point is a boundary point of the set. It is important to note that all boundary points of a convex set are not necessarily extreme points.

Convex Hull

The convex hull C(X) of any given set of points X is the set of all convex combinations of sets of points from X.

Example : If X is the set of eight vertices of a cube, then the convex hull C(X) is the whole cube.

Convex Function

A function f(x) is said to be strictly convex at X if for any two other distinct points x_1 and x_2,

$$f[\lambda x_1 + (1-\lambda)\, x_2] < [f(\lambda x_1) + f(1-\lambda)\, x_2] \text{ where } 0 < X < 1$$

On the other hand, a function f(x) is strictly concave if -f(X) is strictly convex.

Convex Polyhedron

The set of all convex combinations of finite numbers of points is called Convex Polyhedron generated by these points.

19.4 SOME THEOREMS ON CONVEX SETS

Theorem 1 : A hyperplane is a convex set

Proof: Let us consider a hyperplane

Hyperplane

The equation of a hyperplane is H = {x: c x = z }

Let x_1 *and* x_2 be any two points in the

$$\therefore \quad c_1x_1 = z \quad \text{and} \quad c_2x_2 = z$$

If

$$X_3 = [\lambda x_1 + (1-\lambda)\, x_2] \qquad 0 \le \lambda \le 1$$

Then

$$CX_3 = [\lambda C x_1 + (1-\lambda)\, Cx_2]$$

$$= \lambda Z + (1-\lambda)\, Z = Z$$

It implies that X_3 is also a point in the hyperplane X. Hence by definition hyperplane x is a convex set.

Theorem 2: Intersection of two convex sets is also a convex set.

Proof: Let us consider two convex sets X_1 and X_2 and further suppose that X_3 denotes the intersection of these two sets i.e.,

$$X_3 = X_1 \cap X_2$$

If $\quad x_1 = X_1 \cap X_2 \Rightarrow x_1 \in X_1 \text{ and } x_1 \in X_2$

$\quad x_2 = X_1 \cap X_2 \Rightarrow x_2 \in X_1 \text{ and } x_2 \in X_2$

Since are convex sets

$$x_1, x_2 \in X_1 \quad \Rightarrow \quad [\lambda x_1 + (1-\lambda)\, x_2] \in X_1 \; 0 \le \lambda \le 1$$

And $\quad x_1, x_2 \in X_2 \quad \Rightarrow \quad [\lambda x_1 + (1-\lambda)\, x_2] \in X_2 \; 0 \le \lambda \le 1$

Now $\quad [\lambda x_1 + (1-\lambda)\, x_2] \in X_1$ and $[\lambda x_1 + (1-\lambda)\, x_2] \in X_2$

$$\Rightarrow \quad [\lambda x_1 + (1-\lambda)\, x_2] \in X_1 \cap X_2, \; 0 \le \lambda \le 1$$

Hence Intersection of two convex ($X_3 = X_1 \cap X_2$) sets is also a convex set.

Theorem 3: The set of all feasible solutions of a L.P.P is a convex set.

Proof: Let X be the set of all feasible solutions of a L.P.P.

$$Ax = b, \qquad x \ge 0 \qquad \textbf{……….(1)}$$

Case I : If the set X has only one element, then X is convex set. Hence the theorem is true in this case.

Case II : If the set X has atleast two elements.

Let x_1 and x_2 be any two distinct elements in X.

Then $\quad Ax_1 = b, \qquad x_1 \ge 0$

And $\quad Ax_2 = b, \qquad x_2 \ge 0$

If $\quad x_3 = \lambda x_1 + (1-\lambda)\, x_2, \qquad 0 \le \lambda \le 1$

$$Ax_3 = A\lambda x_1 + (1-\lambda)Ax_2$$

$$= \lambda b + (1-\lambda)b = b$$

Also since $x_1 \geq 0, x_2 \geq 0$ $\lambda \geq 0,\ 1-\lambda \geq 0,\ as\ \lambda \leq 1$

$$x_3 = \lambda x_1 + (1-\lambda)\, x_2 \geq 0$$

If x_3 satisfy (1). Thus $x_3 = \lambda x_1 + (1-\lambda)\, x_2 \geq 0$ is also a feasible solution and so belongs to set X.

But x_3 is a convex combination of any two distinct points x_1 and x_2 in X.

Hence by definition the set X is a convex set.

Theorem 4: Every basic feasible solution of the system AX = B, X 0 is an extreme point of the convex set of feasible solutions and conversely.

Corr. 1 : The extreme points of the convex set of feasible solutions are finite in number.

Corr. 2 : An extreme point can have at most m-positive $x_i's$ where m is the number of constraints.

Corr. 3 : In an extreme point, vectors associated to the positive $x_i's$ are linearly independent.

Proof: The proof of the above theorem is left as an exercise for the students.

FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

Theorem 5 : The collection of all feasible solutions to LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.

Proof. Let F be a set of all feasible solutions of the system

$Ax = b,\ x \geq 0.$

If the set F of solutions has only one element, then F is a convex set. Hence the theorem is true in this case.

Now assume that there are at least two distinct points x(1) and x(2) in F.

Then we have

$$Ax_1 = b\,(\,for\,x_1 \geq 0) \quad and \quad Ax_2 = b \qquad (\,for\ x_2 \geq 0) \quad ....(1)$$

We only need to show that every convex combination of any two feasible solutions is also a feasible solution.

We define a new point x(o) as the convex combination of x(1) and x(2) .This implies that

$$x(0) = \lambda x_1 + (1-\lambda)x_2, \qquad 0 \le \lambda \le 1$$

By definition, F is convex if x(o) also belong to F. To show this is true we must show that x(o) satisfies the system of constraints

$Ax = b,\ x \geq 0$

Thus $\quad Ax(o) = A[\lambda x_1 + (1-\lambda)\, x_2]$

$$= \lambda Ax_1 + A(1-\lambda)\, x^2$$

$$= \lambda b + (1-\lambda)b = b$$

Also since $0 \le \lambda \le 1,\ x_1 \ge 0$ and $x_2 \ge 0$ then x(o) ≥ 0.,

This means that x(o) $\subset$ F and consequently F is convex

Theorem 6 : If the convex set of the feasible solutions of AX = B, $X \ge 0$ is a convex polyhedron, then at least one of the extreme point gives an optimum solution.

Proof: We know that the extreme points of the convex set of feasible solutions of

AX = B, $X \ge 0$ are finite in number.

Let $x_{1,}\ x_2$,........., x_k be the extreme points of the set X of all the feasible solutions of

AX = B, $X \ge 0$.

Let Z be the objective function which is to be maximized be given by Z=CX

If $x^* \in X$ is the optimal solution, then

Max. Z = CX*.

Now, if X* is an extreme point, then the theorem is proved..

Now if X* is not an extreme point in X. Then since X is convex polyhedron,therefore,

X* can be expressed as a convex combination of the extreme point of X,

i.e.

$$= \sum_{i=1}^{n} \lambda_i x_i, \quad \lambda_i \geq 0 \qquad and \qquad \sum \lambda_i = 1$$

$$Z^* = CX^* = c(\lambda_1 x + \lambda_2 x_2 + \dots\dots\dots + \lambda_k x_k)$$

$$= \lambda_1 cx_1 + \lambda_2 cx_2 + \dots\dots\dots + \lambda_k cx_k$$

If maximum of CX_i is CX_p, then

$$Z^* \leq (\lambda_1 + \lambda_2 + \dots\dots\dots + \lambda_k) CX_p$$

Or $Z^* \leq CX_p$

But Z* is the maximum value of Z.

Therefore,

$$Z^* = CX_p, \text{ or } CX^* = CX_p$$

i.e., $X^* = X_p$ (one of the extreme points)

Hence the optimal solution is attained at the extreme point, which proves the theorem.

Theorem 7 : If the objective function of a L.P.P. assumes its optimal value at more than one extreme point, then every convex combination of these extreme points gives the optimal value of the objective function.

Proof: Let us consider the L.P.P.. as follows

Max Z = CX

s.t. AX = B, $X \geq 0$

Let $x_{1,}\ x_2, \dots\dots\dots, x_k$ be the extreme points of the feasible region. If the objective function Z assumes its optimal value Z* at the extreme points $x_{1,}\ x_2, \dots\dots\dots, x_p$, $p \leq k$ then

$$Z^* = cx_1 + cx_2 + \dots\dots\dots + cx_k$$

If x_0 is the convex combination of the extreme points $x_{1,}\ x_2, \dots\dots\dots, x_p$; then

$$x_0 = \lambda_1 x + \lambda_2 x_2 + ...+ \lambda_p x_p, \qquad \lambda_i \geq 0, \sum_{i=1}^{p} \lambda_i$$

$$\therefore \qquad cx_0 = c[\lambda x_1 + \lambda_2 x_2 + ...+ \lambda_p x_p]$$

$$= \lambda_1 cx_1 + \lambda_2 cx_2 + ...+ \lambda_p cx_p$$

$$= \lambda_1 Z^* + \lambda_2 Z^* + ...+ \lambda_p Z^*$$

$$= (\lambda_1 + \lambda_2 + ...+ \lambda_p) Z^*$$

$$= Z^* \qquad \left[\because \sum_{i=1}^{p} \lambda_i = 1\right]$$

Hence the optimal value Z* is also attained at x_0 which is the convex combination of the extreme points at which optimal value occurs. Hence the theorem.

19.3 Some Illustrations

Find all the basic feasible solutions for the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

$$x_i \geq 0$$

Also determine the associated general convex combination of the extreme point solutions.

Solution: In matrix form the given system of equations can be written as

AX=B

where $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and

The problem can have at most ${}^4C_2 = 6$ basic solutions.

The six sets of two vectors are

$$B_1 = (\alpha_1, \alpha_2) = \begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}, \quad B_2 = (\alpha_1, \alpha_3) = \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix},$$

$$B_3 = (\alpha_1, \alpha_4) = \begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}, \quad B_4 = (\alpha_2, \alpha_3) = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix},$$

$$B_5 = (\alpha_2, \alpha_4) = \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}, \quad B_6 = (\alpha_3, \alpha_4) = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix},$$

Here $|B_1| = -28, \quad |B_2| = -4, \quad |B_3| = 6, \quad |B_4| = 16, \quad |B_5| = 32, \quad |B_6| = 8,$

Since none of these is zero, therefore all these sets are linearly independent. Hence all the six basic solutions exist.

If X_{B_i} ., $i = 1, 2, \ldots, 6$ are two vectors of the basic variables associated to the sets B_i, $i = 1, 2$.,. 6, respectively, then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X_{B1} = B_1^{-1} B = -\frac{1}{28} \begin{pmatrix} 4 & -6 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

Similarly

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7/2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -7/3 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Thus it is obvious that out of these only three basic solutions are basic feasible solutions (in which variables are non-negative). But the basic feasible solution corresponds to the extreme points. Hence the only three extreme point solutions are given by

$x_1 = (0, 1/2, 0, 0)$, $x_2 = (0, 1/2, 0, 0)$, $x_3 = (0, 1/2, 0, 0)$

$x_1 = x_2 = x_3$. Hence there is a unique extreme point solution.

19.3 SELF ASSESSMENT QUESTIONS

1. Prove that the objective function-of L.P.P assumes its minimum value at an extreme point of the convex set X generated by the set of all feasible solution.

2. Prove that the collection of all feasible solutions to L P. P constitutes a convex set whose extreme points correspond to the basic feasible solutions.

3. Which of the following sets are convex; if so why?

   (i) $X = \{(x_1, x_2) : x_1 + x_2 \le 1, x_1 \ge 0, x_2 \ge 0\}$

   (ii) $X = \{(x_1, x_2) : x_1 + x_2^2 \le 3, x_1 \ge 0, x_2 \ge 0\}$

   (iii) $X = \{(x_1, x_2)\, x_1 \ge 2, x_2 \le 3\}$

4. Define the following terms

   (i) Convex sets and convex combination (ii) Hyperplane and Hypersphere

   (iii) Convex Polyhedron and Convex Hull

19.4 SUMMARY AND FURTHER SUGGESTED READING

- **Convex Set**: A set of points is said to be convex if for any two points in the set, the line segment joining these points is also in the set.
- A hyperplane is a convex set.
- Intersection of two convex sets is also a convex set.
- The set of all feasible solutions of a L.P.P is a convex set.
- If the convex set of the feasible solutions of AX = B, $X \ge 0$ is a convex polyhedron, then atleast one of the extreme point gives an optimum solution.
- If the objective function of a L.P.P assumes its optimal value at more than one extreme point, then every convex combination of these extreme points gives the optimal value of the objective function.

FURTHER SUGGESTED READING

1. Kanti Swarup, Gupta and Man Mohan: Operations Research, Sultan Chand & Sons.

2. V.K. Kapoor (2001) Operations Research. Sultan Chand & Sons

3. R.K. Gupta Linear proramming, Krishna Prakashan Mandir Publishers, Meerut.

4. Mital and Sahni Linear Programming,Pragati Parkashan,Meeru

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20.1 OBJECTIVES

The main Objectives of this lesson are:—

- To define and elaborate Linear Programming Problem.
- Solve a Linear Programming Problem using simplex procedure
- Know about theoretical results concerning Linear Programming Problem
- Understand basic terminology for solution of Linear Programming problem.

20.2 INTRODUCTION TO SIMPLEX METHOD AND SOME BASIC TERMS IN THIS METHOD

In large sized linear programming problems, the solution cannot be obtained by the graphical method and hence a more systematic method has to be developed to find the optimal solution. The 'Simplex Method' developed by George B. Dantzig(1947) is an efficient algorithm to solve such problems. The simplex method is an iterative procedure for moving from an extreme point with a low profit value to another with a higher profit value until the maximum value of the objective function is achieved.

Simplex algorithm is an iterative (step by step) procedure in which we proceed in a systematic manner from an initial basic feasible solution of a linear programming problem to other basic feasible solutions and finally, in a finite number of steps, to an optimal solution, in such way that the value of the objective function at each step is better than at the previous steps. The simplex procedure can be applied to any problem that can be formulated in the terms of a Linear programming problem.

- **THE STANDARD FORM OF LINEAR PROGRAMMING PROBLEM**.

The characteristics of the standard form are as follows:

1. All constraints are equations except for the non-negativity restrictions which remain inequalities $(X \geq 0)$.

2. The right hand side element of each constraint equation is non-negative.

3. All variables are non-negative.

4. The objective function is of the maximization or the minimization type.

Inequality constraints can be changed to equations by adding to or subtracting from, the left hand side of each such constraint a non-negative variable. These new variables are called as slack variables (if added to less than $\leq$ inequations) and surplus variables (if subtracted from greater than $\geq$ inequations).

Symbolically: Let $X', c \in R^n$ and $z = cx$ be a linear function on R^n. Let A be an $m \times n$ real matrix of rank in. Then, the problem of determining x, so as to

$$z = cx$$
$$Ax = b$$
$$x \geq 0$$

Where b is a $m \times 1$ real matrix, is said to be a Linear Programming Problem in standard form.

- **SLACK VARIABLES**

If a constraint has $\leq$ sign, then in order to make it equality we have to add something positive to the left hand side.

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

For example, consider the constraints.

$$x_1 + x_2 \leq 6,$$
$$2x_1 + 5x_2 \leq 8,$$
$$x_1, x_2 \geq 0$$

We add the slack variables $s_3 \geq 0, \; s_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$x_1 + x_2 + s_3 = 6$$

$$2x_1 + 5x_2 + s_4 = 8$$

$$x_1, x_2, s_3, s_4 \geq 0$$

- **SURPLUS VARIABLES**

If a constraint has $\geq$ sign, then in order to make it on equality we have to subtract something non-negative from its left hand side.

Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is culled the surplus variable.

For example, consider the constraints

$$x_1 + x_2 \geq 6,$$
$$2x_1 + 5x_2 \geq 12,$$
$$x_1, x_2 \geq 0$$

We subtract the surplus variables $s_3 \geq 0$, s_4 0 on the left hand sides of above inequalities respectively to obtain

- **BASIC SOLUTION**

For a system of m simultaneous Linear equations in n variables $(n > m)$, a solution obtained by setting $(n - n)$ variables equals to zero and solving for the remaining rn variables is called a basic solution. The $(n - n)$ variables set equal to zero many solution are called non-basic variables.

A solution obtained after setting exactly (n-m) variables to zero, provided the determinant formed by the columns associated to the remaining m variables, is non-zero, is called a Basic Solution.

Basic solutions play a very important role in the theory of simplex method. The superiority of these solutions is that they are finite in number. In m vectors out of n can be selected in

$$^{n}C_{m} = \frac{n!}{m!(n-m)!}$$

ways and hence ${}^{n}C_{m}$ is the upper bound for the number of basic solutions.

- **BASIC FEASIBLE SOLUTION**

A basic feasible solution to a Linear Programming Problem is a basic solution for which the m variables solved for, are all greater than or equal to zero. In other words, a basic solution which happens to be feasible is called a basic feasible solution.

The basic Solutions are of two types:

(a) Non-degenerate Basic Solution — A basic solution is called non-degenerate basic solution if none of the basic variables is zero. In other words all the m basic variables are non-zero.

(b) Degenerate Basic Solution: A basic solution is called degenerate basic solution if one or more of the basic variables are zero.

- **OPTIMUM SOLUTION**

A feasible solution to a L.P. problem is said to be optimum (or optimal) if it also optimizes the objective function Z of the problem.

20.1 THE SIMPLEX PROCEDURE

The Simplex Method is an iterative procedure which either solves a L.P.P. in a finite number of steps or gives an indication that there is an unbounded solution to the L PP. It will now be introduced and explained. In the coming section theory behind the method will first be discussed and then the Computational techniques explained and discussed with the help of sample problems.

The Computational Procedure: The optimal solution to a General L.P.P. (when it exists) is obtained in the following major steps

Step 1.Select an initial (starting) basic feasible solution to initiate the algorithm.

Step 2. Check the objective function to see whether there is some non-basic variable that would improve the objective function if brought in the basis. If such a variable exists, go to step 3, otherwise stop.

Step 3. Determine how large the variable found in step 2 can be made until one of the basic variables in the current solution becomes zero. Eliminate the latter variable and let the next trial solution contain the newly found variable instead.

Step 4. Check for optimality of the current solution.

Step 5. Continue the iterations until either an optimum solution is attained or there is an indication that an unbounded solution exists.

Algorithm: For the solution of any L.P.P. by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

Step 1. Check whether the objective function of the given L.P.P. is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result

$$\text{Minimum } z = - \text{Maximum } (-z).$$

Step 2. Check whether all $b_i \ (i = 1, 2, \ldots, m)$ are non-negative. If any one of b_i is negative, then multiply the corresponding inequality of the constraints by -1, so as to get all $b_i \ (i = 1, 2, \ldots, m)$ non-negative.

Step 3. Convert all the inequations of the constraints into equations by introducing slack /or surplus variables in the constraints. Put the costs of these variables equal to zero.

Step 4. Obtain an initial basic feasible solution to the problem in the form $x_B = B^{-1} b$ and put it in the first column of the simplex table.

Step 5. Compute the net evaluations $z_j - C_j \ (j = 1, 2, \ldots, n)$ by using the relation

$$z_j - c_j = c_B . y_j - c_j$$

Examine the sign of $z_j - c_j$.

(i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution X_B is an optimum basic feasible solution.

(ii) If at least one $(z_j - c_j) < 0$, proceed on to the next step.

Step 6. If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_r - c_r$,. for some $j = r$.

(i) If all $y_{ir} \leq 0$, $(i 1, 2, ..., m)$, then there is an unbounded solution to the given problem.

(ii) If at least one $y_{ir} > 0$, $(i = 1,2, ..., m)$ then the corresponding vector y_r enters the basis y_B.

Step 7. Compute the ratios $\left\{ \frac{x_{Bi}}{y_{ir}},\ y_{ir} > 0,\ \ i = 1,2,....m \right\}$ and choose the minimum of them. Let the minimum of these ratios be $\frac{x_{Bk}}{y_{kr}}$, Then the vector y_k will leave the basis y_B.

The common element y_{kr} , which is in the kth row and the rth column is known as the leaving element (or *pivotal element*) of the table.

Step 8. Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeros.

Step 9. Go to Step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of n unbounded solution.

20.1 SOME FUNDAMENTAL THEOREMS ON SIMPLEX

Theorem 1: If a Linear Programming Problem

$$
\begin{aligned}
&\text{Max } Z = CX, \\
&\text{subject to} \\
&\quad AX = b, \\
&\quad\ X \geq O,
\end{aligned}
$$

where A is a $m \times n$ matrix of coefficients has an optimal solution, then atleast one basic feasible solution must be optimal.

Theorem 2 : If a Linear Programming Problem

$$\text{Max } Z = CX,$$
subject to
$$AX = b,$$
$$X \geq O,$$

where A is a mxn matrix of coefficients has at least one feasible solution, then it has atleast one basic feasible solution also.

20.5 ILLUSTRATION

Illustration: Use simplex method to solve the following L.P.P.

$$\text{Maximize } z = 7x_1 + 5x_2$$
subject to the constraints;
$$x_1 + 2x_2 \leq 6$$
$$4x_1 + 3x_2 \leq 12$$
$$x_1, x_2 \geq 0.$$

Solution: We observe that the given L.P.P. is that of maximizing the objective function subject to the given constraints in which the upper limits are non-negative.

The inequations of the constraints can be converted into equations

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad\quad &= 6 \\ 4x_1 + 3x_2 \quad\quad + x_4 &= 12 \end{aligned} \qquad \text{……..(1)}$$

by introducing slack variables x_3 and x_4 respectively.

Then the objective function is

$$\text{Maximize } z = 7x_1 + 5x_2 + 0.x_3 + 0.x_4$$

The set of equation (1) can be written as

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

$AX = b$

Clearly $\text{Rank}(A) = 2$ and since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent column vectors of A, we can take $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as a non-singular basis sub-matrix of A. The basic variables are, therefore, x_3 and x_4 and an obvious basic feasible solution is

$$x_B = B^{-1} b$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

i.e., $$x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

Corresponding to theses basic variables, the matrix $Y = B^{-1}A$

& the net evaluations

$$z_j - c_j = c_B . y_j - c_j \ ; \ j = 1,2.....n$$

are now computed, where C_B is the matrix of costs corresponding to the basic variables in the objective function. Now, for this initial basic feasible solution

$$z = c_B . x_B \qquad \text{(since the remaining } x_i\text{'s are zero.)}$$
$$= \sum_{i=3,4} c_{Bi} . x_{Bi}$$
$$= 0$$

To see whether there exists some better basic feasible solution, we compute $z_j - c_j$ for the non-basic variables x_1 and x_2 as follows:

$$z_1 - c_1 = c_B . y_1 - c_1$$

$$= (0,0)\begin{bmatrix}1\\4\end{bmatrix} - 7 = -7$$

$$z_2 - c_2 = c_B.y_2 - c_2$$

$$= (0,0)\begin{bmatrix}2\\3\end{bmatrix} - 5$$

$$= -5$$

Thus the initial simplex table is

c_B	y_B		C= 7	5	0	0	Min Ratio
		x_B	y_1	y_2	y_3	y_4	
0	y_3	$x_3 = 6$	1	2	1	0	$x_{B1}/y_{11} = 6/1 = 6$
0	y_4	$x_4 = 12$	4*	3	0	1	$x_{B2}/y_{i2} = 12/4 = 3$
		$z = 0$	-7	-5	0	0	$z_j - c_j$

Now, since more than one $z_j - c_j$ are negative, therefore we choose the most negative of these, viz., - 7, which lies in the column y_1. Since all the components of y_1 are positive, therefore the vector will enter the basis.

To select the vector which should leave the basis , we compute

$$\left\{\frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0,\ \ i = 1,2\right\}$$

and choose the minimum of these ratios, viz, $\frac{12}{4} = 3$.

Thus the vector y_4 leaves the basis. The leading Common element is 4, which becomes the leading element for the next iteration. The leading element has been shown in the simplex table with a star.

FIRST ITERATION. Introduce y_1 and drop y_4 from y_B. Convert the leading element to unity by dividing that row by 4 and all other members of the column y_1 to zero by using the relations given in Step 8(computational algorithm). Compute again the net evaluations $z_j - c_j$.

Thus the next simplex table is obtained as follows

c_B	y_B	x_B	c= 7 y_1	5 y_2	0 y_3	0 y_4	
0	y_3	$x_3 = 3$	0	5/4	1	-1/4	
7	y_1	$x_1 = 3$	1	3/4	0	1/4	
		$z = 21$	0	1/4	0	7/4	$z_j - c_j$

It is apparent from the table that all the newly computed $z_j - c_j$ are non negative and hence an optimum solution has been obtained. Thus optimum feasible solution to the LP is $x_1 = 3,\ x_2 = 0$; max $Z = 21$

20.6 ARTIFICIAL VARIABLE TECHNIQUES

We can solve any L.P.P. by simplex algorithm. We have seen that if an initial basic feasible solution to the problem is easily identifiable, we arrive at the initial simplex table without much labour. Unfortunately in practice, there exists L.P.P. to which an initial basic feasible solution may not be easily determined.

In order to be able to obtain an initial basis matrix easily, a special procedure is available. Whenever it is not obvious to read off the basis matrix, a desirable number of unit column vectors are inserted in the coefficient matrix, say A, of the constraints. This insertion of the unit column vectors in A is done with an obvious intention that it will help in the determination of a basis sub matrix, say B, of A easily.

In L.P.P. some constraints may have the signs $\geq$ or = with all b_i's positive. In such problems we introduce surplus variables in the constraints with signs $\geq$ and =. In these problems we cannot get the starting basic matrix $B = I_m$.

In order to avoid this problem, we add one more variable to each of such constraints. These variables are called 'artificial variables'. As the name implies these variables are fictitious and represent no physical entities. The artificial variable technique is merely a device to get the starting basic feasible solution so that we may proceed with simplex method to get the optimal solution. Such problems can be solved by Two-Phase Method.

Two-Phase Method

Phase-I:First remove the artificial variables from the basis matrix and introduce other variables (non-artificial variables). This part of the solution in which we remove the artificial variables is called Phase I.

Phase-II: After the removal of artificial variables we proceed for the simplex routine called the Phase II. This method has been demonstrated with the help of an example as given below

Illustration: Use two phase simplex method to solve the following L.P.P.

Maximize $z = 3x_1 + 2x_2$
subject to the constraints;
$2x_1 + x_2 \leq 2$
$3x_1 + 4x_2 \geq 24$
$x_1, x_2 \geq 0.$

Solution. Introducing a slack variable $x_3 \geq 0$, surplus variable $x_4 \geq 0$ and an artificial variable $x_5 \geq 0$ in the constraints of the given L.P P., an initial basic feasibl solution to the problem is

$x_B = [2, \ \ 12]$ (x_3 and x_5 basic) with I_2 as basis matrix.

Phase I. assigning a cost -1to the artificial variable x, and cost 0 to all other variables, the objective function of the given L.P.P. becomes

$z^* = 0x_1 + 0x_2 + 0.x_3 + 0.x_4 - 1.x_5$

The auxiliary problem is to maximize z subject to the given constraints. We solve this

L.P.P. by the simplex method.

STARING TABLE. After computing the matrix Y and the net evaluations $z_j - cj$, the initial simplex table is

c_B	y_B	x_B	0 y_1	0 y_2	0 y_3	0 y_4	-1 y_5
0	y_3	2	2	1*	1	0	0
-1	y_5	12	3	4	0	-1	1
		-12	-3	-4	0	1	0

First Iteration: we introduce y_2 and drop y_3

c_B	y_B	x_B	0 y_1	0 y_2	0 y_3	0 y_4	-1 y_5
0	y_2	2	2	1	1	0	0
-1	y_5	4	-5	0	-4	-1	1
		-4	5	0	4	1	0

Since all $z_j - cj \geq 0$ an optimum basic feasible solution to the given L..PP has been attained. But $\max z^* \leq 0$ and the artificial vector y_5 appears in the basis at positive level. Hence the given L.P.P does not possesses any feasible solution.

20.7 SELF ASSESSMENT QUESTIONS

Excercise: (a) Define a general linear programming problem

(b) What do you mean by a L.P.P. ? What are its limitations?

(c) What are the advantages of L.P.P.? What are the limitations of linear

Programming ?

Excercise: Use simplex method to solve the following LP.P.

$$\begin{aligned}
&\text{Maximize } z = 10x_1 + 5x_2\\
&\text{subject to the constraints;}\\
&2x_1 + 4x_2 \le 12\\
&84x_1 + 6x_2 \le 24\\
&x_1, x_2 \ge 0.
\end{aligned}$$

Excercise: Use simplex method to solve the following LP.P.

$$\begin{aligned}
&\text{Maximize } z = 10x_1 + x_2 + 12x_3\\
&\text{subject to the constraints;}\\
&14x_1 + x_2 - 5x_3 \le 5\\
&15x_1 + 0.5x - 6_3 \le 6\\
&\quad 3x_1 - x_2 - x_3 \le 0\\
&x_1, x_2, x_3 \ge 0.
\end{aligned}$$

Excercise: Use simplex method to solve the following LP.P.

$$\begin{aligned}
&\text{Minimize } z = x_1 + x_2\\
&\text{subject to the constraints;}\\
&2x_1 + x_2 \ge 4\\
&x_1 + 7x_2 \ge 7\\
&x_1, x_2 \ge 0.
\end{aligned}$$

20.8 SUMMARY AND FURTHER SUGGESTED READING

The main Objectives of this lesson were to define and elaborate Linear Programming Problem to discuss the techniques of solving a Linear Programming Problem using simplex procedure. To Know about theoretical results concerning Linear Programming Problem. The main findings of this lesson are

- Standard form of LPP is $z = cx$, $Ax = b$, $x \ge 0$
- **SLACK VARIABLES:** If a constraint has $\le$ sign, then in order to make it equality,

the non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

- **SURPLUS VARIABLES:** If a constraint has $\geq$ sign, then in order to make it on equality the non-negative variable which is subtracted from the left hand side of the constraint to convert it into equation is culled the surplus variable.

- **BASIC SOLUTION** : For a system of m simultaneous Linear equations in n variables $(n > m)$, a solution obtained by setting $(n - n)$ variables equals to zero and solving for the remaining m variables is called a basic. In m vectors out of n can be selected in

$^nC_m = \dfrac{n!}{m!(n-m)!}$ ways and hence ^nC_m is the upper bound for the number of basic solutions.

- **BASIC FEASIBLE SOLUTION: A** basic feasible solution to a Linear Programming Problem is a basic solution for which the m variables solved for, are all greater than or equal to zero. In other words, a basic solution which happens to be feasible is called a basic feasible solution.

The basic Solutions are of two types:(a) Non-degenerate Basic Solution

(b) Degenerate Basic

- **OPTIMUM SOLUTION:** A feasible solution to a L.P. problem is said to be optimum (or optimal) if it also optimizes the objective function Z of the problem.

- **ARTIFICIAL VARIABLE TECHNIQUES:** If there exists L.P.P. to which an initial basic feasible solution may not be easily determined. In order to avoid this problem, we add one more variable to each of such constraints. These variables are called 'artificial variables'. As the name implies these variables are fictitious and represent no physical entities.

FURTHER SUGGESTED READING

1. Shanti Swarup Operations Research.
2. S.D. Sharma: Operations Research.
3. Swaroop, Gupta and ManMohan : Operations Research.
4. A.S. Narang : Linear Programming and Decision Making.

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21.1 OBJECTIVES

After studying this lesson the students should be able to

- Solve LPP using Graphical Method.
- Understand Concept of Duality.
- Understand concept and usage of Duality in Linear Programming Problems
- Understand some important results of Duality.

21.2 INTRODUCTION TO GEOMETRICAL METHOD FOR SOLVING L.P.P

The graphical method is a simple one, and is the most easily understood of the several linear programming methods. A thorough knowledge of the graphical procedure provides necessary insight and confidence to understand the more advanced methods and concepts behind these methods. However, it should be pointed out that the graphical method can be applied only in the case of two variables. It cannot be applied to problems with many variables.

21.3 PROCEDURE FOR GEOMETRICAL METHOD FOR SOLVING L.P.P

If z is a function of two variables then the problem can be solved graphically. Solution of a problem of two variables will represent a point in two-dimensional plane. Thus by solving the problem, we mean to find a point in a plane which satisfies all the constraints and gives the optimum value of z. In graphical methods we consider the constraints as equalities and then draw the lines in two dimensional plane corresponding to each equation and non-negativity restrictions. These lines border the region of permissible values of the variables called **feasible region**. It is the region which satisfies all the constraints simultaneously. This region is taken as shaded region.

Next we find a point in the permissible region, which gives the optimum value of z(maximum or minimum value as desired in the problem). To find this point choose a suitable numerical value of z and draw a line corresponding to this value of z known as objective function line. This objective function line of z moved parallel to itself above or below (as the case may be) until an extreme (corner) point of the feasible region is reached and beyond which there exists no point of the feasible region on the line of z. This point represents the **optimal solution**. The optimal values of the variables either can be read from the figure or can be obtained by solving simultaneously the equations of the two lines on which this point lies. The main drawback of this method is that the problem of higher dimensions (more than two dimensions) can not be solved by this method. A problem of three dimensions can also be handled by this method but it is quite complicated.

ILLUSTRATION: Let us consider a small foundry which specializes in the production of iron castings. For the sake of simplicity, assume that the foundry specializes in producing two types of castings type A and type B. It is assumed that the foundry can sell as many units as it produces. The profit is Rs.70 and Rs.40 for each of casting A and casting B respectively. The foundry manager should decide the quantity of these castings to be produced each week so as to maximize the total profit.

Production of castings requires certain resources like raw materials, labor and foundry capacity and the constraints on these resources are given by

Resources	Requirement per unit of		Availability in a week
	type A	type B.	
Raw material-I	2Kgs.	1Kg	120 Kgs
Raw material-II	0.8 Kgs	None	40 kgs
Labour	3 man-days	2 man-days	200 man days
Foundry capacity	4 units	3units	360 units

Solve the given L.P.P by graphic method

Solution: Let $q_1 =$ number of type A castings to be produced

$q_2 =$ number of type B castings to be produced

After mathematical formulation of the above stated problem it can be represented as given below

Maximize $Z = 70q_1 + 40q_2$ ......(1)

Subject to constraints:

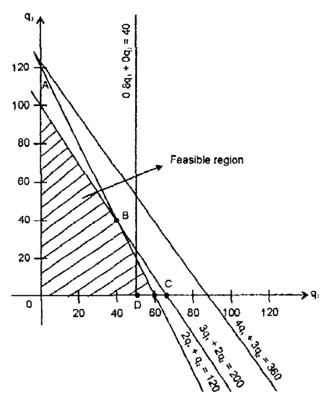
$$\left.\begin{array}{l} 2q_1 + 1q_2 \le 120 \\ 0.8q_1 + 0q_2 \le 40 \\ 3q_1 + 2q_2 \le 200 \\ 4q_1 + 3q_2 \le 360 \end{array}\right\} \qquad \text{.......(2)}$$

$$q_1 \ge 0, \qquad q_2 \ge 0 \qquad \text{......(3)}$$

Solution: We have to find the values of q_1 and q_2 which will satisfy constraints (2) and (3) and at the same time maximize function (1). The function given in (1) is called an objective function. The inequalities in (2) are called constraints and the inequalities in (3) are called non-negativity restrictions or constraints. This problem cannot be solved by the calculus method because of the inequality constraints.

The first step in the graphical method of solution is to identify the region in the graph which corresponds to all pairs of values of q_1 and q_2 for which (2) and (3) are valid.

Let us consider the non-negativity restrictions given by (3). The values of which satisfy these restrictions should fall in the first quadrant of the graph. Hence, we can ignore pairs of values of which fall in other quadrants. This is indicated by arrow marks on the (or x-axis) and (or y-axis) in the graph shown below



Let us next find the region corresponding to the values of q_1 and q_2 for which the first constraint

For this, we need to fix two points on this line. The points that we have chosen are:

$$2q_1 + 1q_2 \leq 120$$

is satisfied. To do this, we have to first draw the line

$$2q_1 + 1q_2 = 120$$

For this, we need to fix two points on this line. The points that we have chosen are:

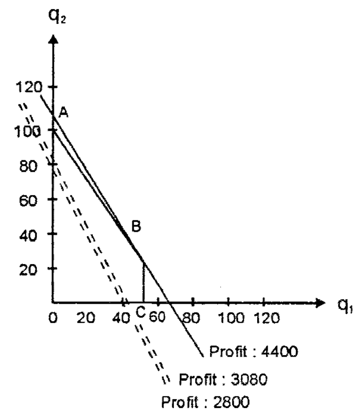
$$q_1 = 0 \qquad q_2 = 120 \qquad \textit{and}$$
$$q_1 = 60 \qquad q_2 = 0$$

By joining these points, we get the line, and the points below the line, indicated by arrows, will satisfy the first constraint.

Other constraint equations are also drawn on the graph. The region common to all the regions identified gives the set of points for which the values of the co-ordinates satisfy constraints (2) and (3). The region identified is OABCD is called the feasible region. It may be noted that all values of which satisfy constraints (2) and (3), lie within the region OABCD and all points in the region OABCD will have the co-ordinates, which will satisfy constraints (2) and (3). Hence, an optimal solution to the problem should have co-ordinates within or on the boundary of region OABCD.

Now let us search for the optimal solution.

Suppose, we are interested in finding a product mix which will give a profit of say, an arbitrarily chosen value of Rs.2,800. To get the product mix, we have to search in the region OABCD to examine whether any point gives a profit of Rs.2,800. The easiest way is to draw the straight line whose equation is and examine whether it passes through any points in the region OABCD. In the graph given below, the feasible region OABCD and the above mentioned straight line are shown. We can observe that there are many points on this straight line which come under the feasible region, and each point will give the co-ordinates which refer to the production levels that yield the same profit of Rs.2,800. For example, take the two points (40,0) and (0,70) on this line. The production levels corresponding to these points are (i) 40 of type A castings and 0 of type B castings and (ii) 0 of A castings and 70 of type B castings. It can be verified that each co-ordinate gives the same profit. Thus, the straight line drawn is also the profit line.



Suppose we wish to increase the profit, we look for a product mix which will give a profit of, say, Rs.3,080. As done earlier, we draw the line

$$70q_1 + 40q_2 = 3{,}080$$

and examine whether it passes through the region OABC. This line is parallel to the first line and passes through the feasible region, thus indicating that it is possible to increase the profit to Rs.3,080. This suggests that, as we move up this line in the Northeastern direction, parallel to itself, we can obtain product mixes which will give higher and higher profits. We should move the line as far as possible without removing it completely from the region of feasible solutions as otherwise we will not find any feasible product mix which will satisfy the constraints. The optimal solution is then given by the point of final contact, which will be one of the corner points. In this case, the point is B, whose co-ordinates are (40, 40), indicating that the production level should be 40 for each of type A and B castings and this will yield a profit of

21.2 DUALITY IN L.P.P

For every LP formulation there exists another unique linear programming formulation called the ‘Dual’ (the original formulation is called the ‘Primal’). Same data can be used for both ‘Dual’ and Primal’ formulation. Both can be solved in a similar manner as the Dual is also an Linear Programming formulation.

The Dual can be considered as the ‘inverse’ of the Primal in every respect. The column coefficients in the Primal constraints become the row co-efficients in the Dual constraints. The coefficients in the Primal objective function become the right-hand-side constraints in the Dual constraints. The column of constants on the right hand side of the Primal constraints becomes the row of coefficients of the dual objective function. The direction of the inequalities are reversed. If the primal objective function is a ‘Maximization’ function then the dual objective function is a ‘Minimization’ function and vice versa.

A symmetric relation between primal and its dual problem exists.Let us consider a linear programming problem

$$Max\, z_p = c_1x_1 + c_2x_2 + ...+ c_nx_n$$

$$Subject\ to$$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n \le b_2 \\ ------------------ \\ a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n \le b_m \end{array}\right] \qquad ....(1)$$

$$x_1, x_2, ..., x_n \ge 0$$

The dual problem of the above L.P. problem is obtained by

(i) Transposing the coefficient matrix.

(ii) Interchanging the role of constant terms and the coefficients of the objective function.

(iii) Reverting the inequalities and

(iv) Minimizing the objective function instead of maximizing it. The dual problem is as follows:

Find $w_1, w_2, ...w_m$

$$
\begin{aligned}
&Min\, z_D = b_1 w_1 + b_2 w_2 + ...+ b_m w_m \\
&Subject\ to \\
&\left.\begin{aligned}
&a_{11} w_1 + a_{21} w_2 + ... + a_{m1} w_m \geq c_1 \\
&a_{12} w_1 + a_{22} w_2 + ...+ a_{m2} w_m \geq c_2 \\
&------------------ \\
&a_{1n} w_1 + a_{2n} w_2 + ...+ a_{mn} w_m \geq c_n
\end{aligned}\right] \qquad \ldots\ldots\ldots\ldots\ldots(2) \\
&w_1, w_2, ..., w_n \geq 0
\end{aligned}
$$

If a system consists of a mixture of equations, inequalities (in either direction), non-negative variables or unrestricted variables then the dual of the problem can be obtained by reducing to the form (1). By using the following procedure:

(i) If a constraint has a sign $\geq$, then multiply both sides by-1 and make sign $\leq$.

(ii) If a constraint has a sign =, , then it is replaced by two constraints involving the inequalities going on opposite directions i.e. $\leq$ and $\geq$ and then multiply both sides of by -1.

(iii) Every unrestricted variable is replaced by the difference of two positive variables.

Why the Dual Formulation?

Dual formulation is done for a number of reasons. The solution to a Dual problem provides all essential information about the solution to the Primal problem. A solution for the LP problem can be determined either by solving the original problem or the Dual problem. Sometimes it may be easier to solve the Dual problem rather than the Primal problem as when the primal involves few variables but many constraints.

Illustration: Consider the following 'Primal' LP formulation.

$$
\begin{aligned}
&\text{Maximize}\, 12x_1 + 10x_2 \\
&Subject\ to \\
&2x_1 + 3x_2 \leq 18 \\
&2x_1 + x_2 \leq 14 \\
&x_1, x_2 \geq 0
\end{aligned}
$$

The 'Dual' formulation for this problem would be

$$\begin{aligned}&\text{Minimize } 18y_1 + 14y_2\\&\text{subject to}\\&2y_1 + 2y_2 \geq 12\\&3y_1 + y_2 \geq 10\\&y_1, y_2 \geq 0\end{aligned}$$

1. The column coefficient in the Primal constraint namely (2,2) and (3,1) have become the row coefficient in the Dual constraints.
2. The coefficient of the Primal objective function namely, 12 and 10 have become the constants in the right hand-side of the Dual constraints.
3. The constants of the Primal constraints, namely 18 and 14, have become the coefficient in the Dual objective function.
4. The direction of the inequalities have been reversed. The Primal constraints have the inequalities of while the Dual constraints have the inequalities of .
5. While the Primal is a 'Maximization' problem the Dual is a 'Minimization' problem and vice versa.

21.2 SOME THEOREMS ON DUALITY

Theorem 1: The dual of the dual of a given primal is the primal.

Proof. Let the primal problem is given by

Primal.

$$\begin{aligned}&Max\, z_x = c_1x_1 + c_2x_2 + ...+ c_nx_n\\&Subject\ to\\&\left.\begin{aligned}&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1\\&a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n \leq b_2\\&------------------\\&a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n \leq b_m\end{aligned}\right] \qquad ....(1)\\&x_1, x_2, ..., x_n \geq 0\end{aligned}$$

The dual problems is

$$
\begin{aligned}
&Min\, z_x = b_1 w_1 + b_2 w_2 + ...+ b_m w_m \\
&Subject\ to \\
&\left.\begin{aligned}
&a_{11} w_1 + a_{21} w_2 + ... + a_{m1} w_m \geq c_1 \\
&a_{12} w_1 + a_{22} w_2 + ...+ a_{m2} w_m \geq c_2 \\
&------------------- \\
&a_{1n} w_1 + a_{2n} w_2 + ...+ a_{mn} w_m \geq c_n
\end{aligned}\right] \qquad ....(2) \\
&w_1, w_2, ..., w_n \geq 0
\end{aligned}
$$

Now, our aim is to construct the dual of the dual (2) . For this, we first express the above dual (2) in the standard form of the primal by multiplying through by -1 Thus, we obtain,

$$
\begin{aligned}
&Max(-z_x) = -b_1 w_1 - b_2 w_2 + ...- b_m w_m \\
&Subject\ to \\
&\left.\begin{aligned}
&-a_{11} w_1 - a_{21} w_2 - .... - a_{m1} w_m \leq -c_1 \\
&-a_{12} w_1 - a_{22} w_2 - ....- a_{m2} w_m \leq -c_2 \\
&------------------- \\
&-a_{1n} w_1 - a_{2n} x_2 - ....- a_{mn} w_n \leq -c_n
\end{aligned}\right] \qquad ....(3) \\
&w_1, w_2, ..., w_n \geq 0
\end{aligned}
$$

Dual of Dual: Now, constructing the dual of the dual (3) we get

$$
\begin{aligned}
&Min\, z_v = -c_1 v_1 - c_2 v_2 - ....- c_n v_n \\
&Subject\ to \\
&\left.\begin{aligned}
&-a_{11} v_1 - a_{12} v_2 - .... - a_{1n} v_n \geq b_1 \\
&-a_{21} v_1 - a_{22} v_2 - ....- a_{2n} v_n \geq -b_2 \\
&------------------- \\
&-a_{m1} v_1 - a_{m2} v_2 - ....- a_{mn} v_n \geq -b_m
\end{aligned}\right] \qquad ....(4) \\
&v_1, v_2, ..., v_n \geq 0
\end{aligned}
$$

Now, multiplying (4) throughout by -1, we get the equivalent system:

$$\begin{aligned}
&Max\, z_v' = c_1v_1 + c_2v_2 + ...+ c_nv_n\\
&Subject\ to\\
&\left.\begin{aligned}
&a_{11}v_1 + a_{12}v_2 + ... + a_{1n}v_n \le b_1\\
&a_{21}v_1 + a_{22}v_2 + ...+ a_{2n}v_n \le b_2\\
&-------------------\\
&a_{m1}v_1 + a_{m2}v_2 + ...+ a_{mn}v_n \le b_m
\end{aligned}\right] \qquad \text{..............(5)}\\
&v_1, v_2, ..., v_n \ge 0
\end{aligned}$$

which is the system identical to the primal (1).This completes the proof of the theorem.

Theorem 2. The necessary and sufficient condition for any L.P.P and its dual to have optimal solution is that both have feasible solutions.

Theorem 3. Fundamental Duality Theorem

If either the primal or the dual problem has finite optimal solution, then the other problem also has a finite optimal solution and the value of the two objective functions are equal.

21.2 SELF ASSESSMENT QUESTIONS

Exercise 1: (a) Describe graphical procedure for solution to the linear programming problem

(b) Define (i) feasible region (ii) Objective function line in reference to the graphical solution to the linear programming problem.

(c) What do you mean by duality in L.P.P.?

Exercise: Describe the role of Dual Formulation in L.P.P.

Excercise: Use graphical method to solve the following L.P.P.

$$\begin{aligned}
&\text{Maximize } z = 10x_1 + 5x_2\\
&\text{subject to the constraints;}\\
&2x_1 + 4x_2 \le 12\\
&84x_1 + 6x_2 \le 24\\
&x_1, x_2 \ge 0.
\end{aligned}$$

Excercise: Use geometrical procedure to solve the following L.P.P.

$$\text{Minimize } z = 10x_1 + x_2 + 12x_3$$
subject to the constraints;
$$14x_1 + x_2 - 5x_3 \geq 5$$
$$15x_1 + 0.5x - 6_3 \geq 6$$
$$3x_1 - x_2 - x_3 \geq 0$$
$$x_1, x_2, x_3 \geq 0.$$

Exercise: Solve graphically,

$$\text{Max. } Z = 5x_1 + 7x_2$$
s.t. $x_1 + x_2 \leq 4$
$$3x_1 + 8x_2 \leq 24$$
$$10x_1 + 7x_2 \leq 35$$
and $x_1, x_2 \geq 0$

Exercise: Prove that the dual of the dual of a given primal is the primal.

Excercise: Write the dual of the following L.P.P.

$$\text{Minimize } z = x_1 + x_2$$
subject to the constraints;
$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0.$$

21.7 SUMMARY AND FURTHER SUGGESTED READING

The major objectives of studying this lesson was to know about Graphical Method and to understand Concept and usage of Duality. The main findings of this lesson are

- The graphical method is a simple one but it cannot be applied to problems with many variables.

- DUALITY IN L.P.P: For every LP formulation there exists another unique linear programming formulation called the 'Dual' (the original formulation is called the 'Primal'). Same data can be used for both 'Dual' and Primal' formulation. Both can be solved in a similar manner as the Dual is also an Linear Programming formulation.
- The solution to a Dual problem provides all essential information about the solution to the Primal problem. A solution for the LP problem can be determined either by solving the original problem or the Dual problem.
- The dual of the dual of a given primal is the primal.
- The necessary and sufficient condition for any L.P.P and its dual to have optimal solution is that both have feasible solutions.
- If either the primal or the dual problem has finite optimal solution, then the other problem also has a finite optimal solution and the value of the two objective functions are equal.

FURTHER SUGGESTED READING

1. Kanti Swarup, Gupta and Man Mohan: Operations Research, Sultan Chand & Sons.
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3. R.K. Gupta Linear proramming, Krishna Prakashan Mandir Publishers, Meerut.
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